All 4-Edge-Connected HHD-Free Graphs are $\mathbb{Z}_3$-Connected*

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Abstract

An undirected graph $G = (V, E)$ is called $\mathbb{Z}_3$-connected if for all $b : V \to \mathbb{Z}_3$ with $\sum_{v \in V} b(v) = 0$, an orientation $D = (V, A)$ of G has a $\mathbb{Z}_3$-valued nowhere-zero flow $f : A \to \mathbb{Z}_3 - \{0\}$ such that $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = b(v)$ for all $v \in V$. We show that all 4-edge-connected HHD-free graphs are $\mathbb{Z}_3$-connected. This extends the result due to Lai (2000), which proves the fact for chordal graphs.

1 Introduction

Let $D = (V, A)$ be a digraph with vertex set V and arc set A. For a vertex $v \in V$, $\delta^+(v)$ (resp., $\delta^-(v)$) denotes the set of arcs leaving (resp., entering) v. Let $b : V \to \mathcal{A}$, where $\mathcal{A}$ denotes an abelian group with identity 0. When $\sum_{v \in V} b(v) = 0$, b is called a *zero-sum function*. For a zero-sum function b, a *nowhere-zero $(\mathcal{A}, b)$-flow* is defined as a function $f : A \to \mathcal{A} - \{0\}$ such that $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = b(v)$ for all $v \in V$. Notice that the existence of nowhere-zero $(\mathcal{A}, b)$-flows only depends on b and the underlying undirected graph $G = (V, E)$ of D because reversing the direction of an arc $e \in A$ can be cancelled by reversing the sign of $f(e)$. Hence in this paper, we discuss nowhere-zero $(\mathcal{A}, b)$-flows for undirected graphs rather than for digraphs.

An undirected graph G is called *$\mathcal{A}$-connected* when it has a nowhere-zero $(\mathcal{A}, b)$-flow for each zero-sum function $b : V \to \mathcal{A}$. Let $\mathbb{Z}_3$ denote the cyclic group of order 3. This paper is concerned with the following conjecture due to Jaeger et al. [2].

Conjecture 1 ([2]). *Every 5-edge-connected undirected graph is $\mathbb{Z}_3$-connected.* □

This conjecture is closely related to the 3-flow conjecture due to Tutte [9], which is a long-standing open problem in graph theory. A nowhere-zero 3-flow is known to be equivalent to a $(\mathcal{A}, b)$-flow when $|\mathcal{A}| = 3$ and $b(v) = 0$ for all $v \in V$. The 3-flow conjecture can be stated as follows.

Conjecture 2 ([9]). *Every 4-edge-connected undirected graph has a nowhere-zero 3-flow.* □

Kochol [4] has shown that if every 5-edge-connected undirected graphs has a nowhere-zero 3-flow, then every 4-edge-connected undirected graphs does. This means that Conjecture 1 implies the 3-flow conjecture. Refer to [8, 10] for more information on the 3-flow conjecture.

Motivated by Conjecture 1, there are several works investigating which graphs are $\mathbb{Z}_3$-connected [6, 7]. For example, Lai [5] has shown that 4-edge-connected chordal graphs are $\mathbb{Z}_3$-connected. A simple proof for his result will be presented in Section 2. Our main contribution in this paper is to show the $\mathbb{Z}_3$-connectivity of HHD-free graphs.

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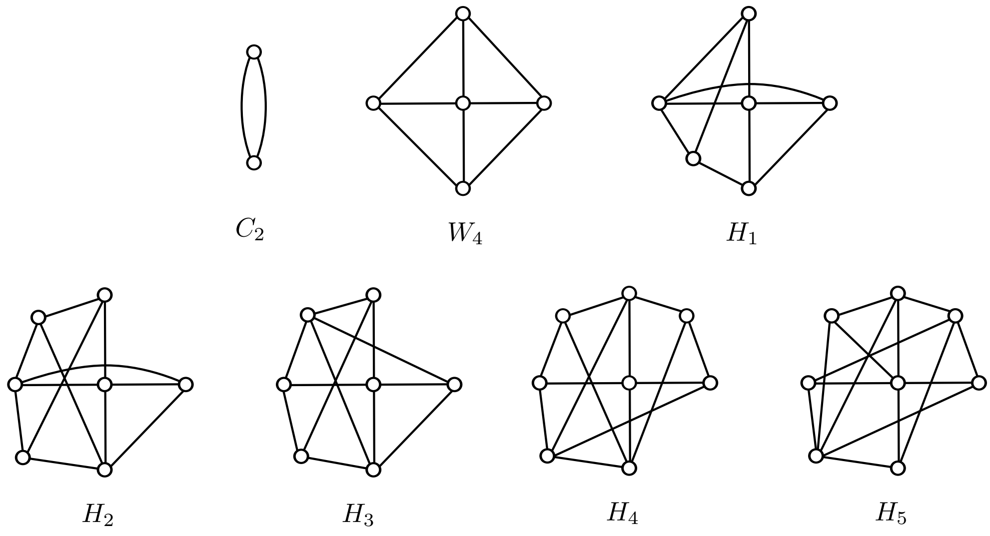

Figure 1: Z_3-connected graphs used for proving Theorem 1

Theorem 1. *Every 4-edge-connected HHD-free graph is $\mathbb{Z}_3$-connected.* □

HHD-free graphs, which will be defined in Section 2, form a super-class of chordal graphs. Thus Theorem 1 is an extension of the result due to Lai [5].

For proving $\mathcal{A}$-connectivity of graphs in a family closed under contraction, it suffices to show that each graph in the family has $\mathcal{A}$-connected subgraphs by the following theorem.

Theorem 2 ([5])**.** *Let H be an $\mathcal{A}$-connected subgraph of an undirected graph G. If $G\backslash H$ is $\mathcal{A}$-connected, then G is also $\mathcal{A}$-connected.* □

Actually most of the previous works on $\mathcal{A}$-connectivity are based on this observation. This paper also takes the same approach with them. We prove the theorem by showing that every non-trivial 4-edge-connected HHD-free graph contains a subgraph isomorphic to one of graphs in Figure 1. Although we omit the proof, All graphs in Figure 1 are $\mathbb{Z}_3$-connected.

Theorem 3. *Graphs C_2, W_4, H_1, $\ldots$, H_5 are $\mathbb{Z}_3$-connected.*

Proof. The proof for C_2 and W_4 can be found in [5]. The Z_3-connectivity of the other graphs can be verified by enumerating all $f : A \to \mathbb{Z}_3 - \{0\}$ for an arbitrary orientation of the graph, which is easy with help of computers. □

2 Preliminaries on Graphs

For an undirected graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. When $E(G) = \{e\}$, $V(G)$ may be represented by $V(e)$. For $u, v \in V(G)$, uv denotes the edge in $E(G)$ joining vertices u and v. For $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. For a vertex v, $N(v)$ denotes the set of neighbors of v.

We let C_k stand for the cycle on k vertices, and P_k stand for the path obtained by removing one edge from C_k. A path P_k with $V(P_k) = \{v_1, \ldots, v_k\}$ and $E(P_k) = \{v_1v_2, \ldots, v_{k-1}v_k\}$ is represented by $v_1 \text{-} \cdots \text{-} v_k$. We let W_k denote a graph obtained by adding one vertex to C_k with edges joining the vertex and each vertex in $V(C_k)$. The *domino* stands for the graph consisting of six vertices and seven edges

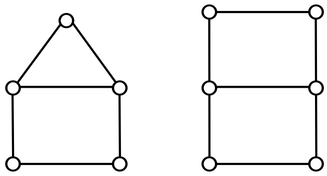

Figure 2: house and domino

where two C_4's share one edge (See Figure 2). The *house* stands for the graph consisting of five vertices and six edges where a C_4 and a C_3 share one edge (see Figure 2).

A graph is called *chordal* if any C_k with $k \geq 4$ has at least one chord, i.e., it contains no induced subgraph isomorphic to C_k with $k \geq 4$. A vertex v is called *simplicial* if $G[N(v)]$ is complete. In other words, it is defined as a vertex which is not a midpoint of any induced P_3. Chordal graphs can be characterized by the existence of simplicial vertices as follows.

Theorem 4 ([1]). *Every chordal graph has a simplicial vertex.* □

We can observe that this characterization presents a simple proof of the theorem due to Lai [5].

Theorem 5 ([5]). *Every 4-edge-connected chordal graph is $\mathbb{Z}_3$-connected.*

Proof. Let G be a counter-example (i.e., G is a 4-edge-connected chordal graph that is not $\mathbb{Z}_3$-connected) minimizing $|V(G)|+|E(G)|$. Let $e, f \in E(G)$ be parallel edges. Then they form a $\mathbb{Z}_3$-connected subgraph C_2, and hence contracting them gives a smaller counter-example by Theorem 2. This contradicts the definition of G. Thus G is simple.

Let v^* be a simplicial vertex in G, whose existence is guaranteed by Theorem 4. Since G is simple and 4-edge-connected, $|N(v^*)| \geq 4$. Moreover, because the subgraph induced by $N(v^*)$ is complete by the definition of v^*, it contains a C_4. Hence the subgraph induced by $N(v^*) \cup \{v^*\}$ contains W_4. Because contracting W_4 gives a smaller counter-example, we have a contradiction. □

A graph is called *HHD-free* if any C_k with $k \geq 5$ has at least two chords. This is equivalent to containing no induced subgraph isomorphic to house, domino, and C_k with $k \geq 5$. Similarly for chordal graphs, HHD-free graphs can be characterized by existence of special vertices. A vertex v is called *semi-simplicial* if v is not a midpoint of any induced P_4.

Theorem 6 ([3]). *Every HHD-free graph has a semi-simplicial vertex.* □

In the next section, we see that this characterization is useful for proving the Z_3-connectivity of 4-edge-connected HHD-free graphs.

3 Proof of Theorem 1

Let G be a 4-edge-connected HHD-free graph. If G is chordal, then G is $\mathbb{Z}_3$-connected by Theorem 5. Hence we assume that G is not chordal. In the following, we show that G always contains a $\mathbb{Z}_3$-connected subgraph.

We can assume without loss of generality that G is simple since otherwise it contains C_2, which is a required subgraph.

Let v^* be a semi-simplicial vertex in G, whose existence is guaranteed by Theorem 6. Let us consider the case where $N(v^*) \cup \{v^*\} = V(G)$. Since G is not chordal, it contains an induced subgraph H

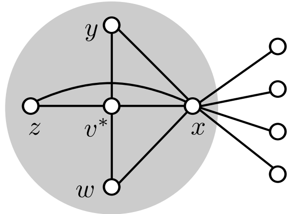

Figure 3: When $|O(\mathcal{P})| = 4$ and $N(\mathcal{P}) = \{x\}$

isomorphic to C_4. The subgraph H does not contain v^* because all vertices in $V(G) - \{v^*\}$ are adjacent to v^*. This means that $G[V(H) \cup \{v^*\}]$ contains a W_4, which is $\mathbb{Z}_3$-connected.

In the remainder of this section, we suppose that $N(v^*) \cup \{v^*\} \neq V(G)$. That is to say, G contains a vertex not in $N(v^*) \cup \{v^*\}$. We let O denote the set of vertices in $V(G) - (N(v^*) \cup \{v^*\})$ that are adjacent to some vertex in $N(v^*)$. By the 4-edge-connectivity of G, there exist at least four edge-disjoint paths between v^* and O. We call a set of edge-disjoint paths $\mathcal{P}$ *minimal* if for each $P \in \mathcal{P}$, G does not contains another path P' between v^* and O such that $V(P') \cap N(v^*) \subset V(P) \cap N(v^*)$ and and P' is edge-disjoint from all paths in $\mathcal{P} - \{P\}$.

Choose a minimal set $\mathcal{P} = \{P_x, P_y, P_z, P_w\}$ of four edge-disjoint paths between v^* and O, where x, y, z and w denote the vertex next to v^* in P_x, P_y, P_z and P_w, respectively. Notice that x, y, z and w are all different vertices because G is simple. For all $v \in \{x, y, z, w\}$, let e_v denote the edge in $E(P_v)$ joining a vertex in $N(v^*)$ and the one in O while $O(e_v)$ denotes $O \cap V(e_v)$ and $N(e_v)$ denotes $N(v^*) \cap V(e_v)$. Moreover, $O(\mathcal{P}) = \{O(e_v) \mid v \in \{x, y, z, w\}\}$ and $N(\mathcal{P}) = \{N(e_v) \mid v \in \{x, y, z, w\}\}$.

When there are several choices of $\mathcal{P}$, we choose one that minimizes $|O(\mathcal{P})|$. If there are two minimal sets $\mathcal{P}' = \{P'_{x'}, P'_{y'}, P'_{z'}, P'_{w'}\}$ and $\mathcal{P}'' = \{P''_{x''}, P''_{y''}, P''_{z''}, P''_{w''}\}$ such that $|O(\mathcal{P}')| = |O(\mathcal{P}'')| = 2$, $O(P'_{x'}) = O(P'_{y'}) = O(P'_{z'}) \neq O(P'_{w'})$ and $O(P''_{x''}) = O(P''_{y''}) \neq O(P''_{z''}) = O(P''_{w''})$, then we give $\mathcal{P}'$ priority over $\mathcal{P}''$.

By the minimality of $\mathcal{P}$, the following observation holds.

Claim 1. $N(\mathcal{P}) \subseteq \{x, y, z, w\}$.

Proof. Suppose that $N(e_x) \notin \{x, y, z, w\}$. Let P' denote the path v^* - $N(e_x)$ - $O(e_x)$. Then $V(P') \subset V(P_x)$ and P' is edge-disjoint from P_y, P_z, and P_w. This contradicts the minimality of $\mathcal{P}$. $\square$

In the following, we classify cases according to $\mathcal{P}$.

When $|O(\mathcal{P})| = 4$

Let $x \in N(\mathcal{P})$ (i.e., e_x is incident with x), and v be a vertex in $N(v^*) - \{x\}$. Then v^* is a midpoint of v - v^* - x - $O(e_x)$. Notice that $O(e_x)$ is not adjacent to v^* since $O(e_x) \notin N(v^*)$, and to v since otherwise one path in $\mathcal{P}$ can be replaced by another path v^* - v - $O(e_x)$, which decreases $|O(\mathcal{P})|$. Hence $E(G)$ must contain an edge xv for forbidding the P_4 from being an induced subgraph.

This fact implies that if $|N(\mathcal{P})| \geq 2$, then $G[\{x, y, z, w\}]$ contains a C_4, and hence $G[\{v^*, x, y, z, w\}]$ contains a W_4. Now let us consider the case of $|N(\mathcal{P})| = 1$. Without loss of generality, let $N(\mathcal{P}) = \{x\}$ (see Figure 3). Each vertex in $N(v^*) - \{x\}$ is adjacent to both v^* and x, and adjacent to no vertex in $V(G) - N(v^*)$ by the minimality of $\mathcal{P}$. From this and the fact that G is 4-edge-connected and simple, the degree of each vertex in $N(v^*) - \{x\}$ is at least 2 in $G[N(v^*) - \{x\}]$. Thus $G[N(v^*)]$ contains a C_4, and hence $G[N(v^*) \cup \{v^*\}]$ contains a W_4.

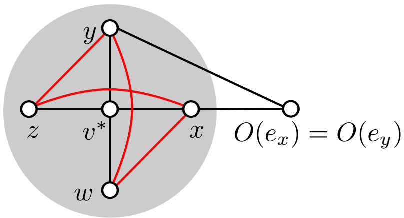

Figure 4: When $|O(\mathcal{P})| = 3$ and $O(e_x) = O(e_y)$

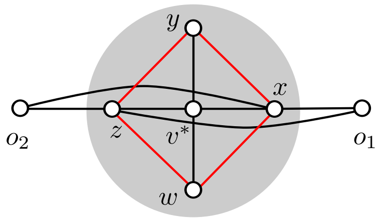

Figure 5: An example with $|O(\mathcal{P})| = 2$, $O(e_x) = O(e_y) \neq O(e_z) = O(e_w)$ and $N(e_x) = N(e_w) \neq N(e_y) = N(e_z)$

When $|O(\mathcal{P})| = 3$

Let $O(e_x) = O(e_y)$ without loss of generality. Since G is simple, $N(e_x) \neq N(e_y)$ would hold, and so $N(e_x) = x$ and $N(e_y) = y$. Let $u \in \{x, y\}$ and $v \in \{z, w\}$. There exists a P_4 v - v^* - u - $O(e_u)$. If an edge $vO(e_u)$ exists, then replacing a path in $\mathcal{P}$ with another path v^* - v - $O(e_u)$ decrease $O(\mathcal{P})$, a contradiction. Thus for forbidding the P_4 from being an induced subgraph, there must be an edge uv. Applying this argument to all pairs of $u \in \{x, y\}$ and $v \in \{z, w\}$, we can see that $G[\{x, y, z, w, v^*\}]$ contains a W_4 (see Figure 4).

When $|O(\mathcal{P})| = 2$ and $O(e_x) = O(e_y) \neq O(e_z) = O(e_w)$

We denote $O(e_x) = O(e_y)$ by o_1, and $O(e_z) = O(e_w)$ by o_2. Since G is simple, $N(e_x) \neq N(e_y)$ and $N(e_z) \neq N(e_w)$. Let $u \in \{N(e_x), N(e_y)\}$ and $v \in \{N(e_z), N(e_w)\}$. Notice that $u = v$ may hold (see an example in Figure 5).

If there exists an edge uo_2 (resp., vo_1), then replacing path P_u (resp., P_v) by another path v^* - u - o_2 gives another set $\mathcal{P}' = \{P'_{x'}, P'_{y'}, P'_{z'}, P'_{w'}\}$ of four edge-disjoint paths between v^* and O such that $|O(\mathcal{P}')| = 2$ and $O(e_{x'}) = O(e_{y'}) = O(e_{z'}) \neq O(e_{w'})$. This contradicts the definition of $\mathcal{P}$. Hence no edge joins u and o_2 (resp., v and o_1). When $u \neq v$, for forbidding a P_4 u - v^* - v - o_2 (resp., v - v^* - u - o_1) from being an induced subgraph, there must be an edge uv. This tells that $G[\{v^*, x, y, z, w\}]$ contains W_4.

When $|O(\mathcal{P})| = 2$ and $O(e_x) \neq O(e_y) = O(e_z) = O(e_w)$

We denote $O(e_x)$ by o_1 and $O(e_y) = O(e_z) = O(e_w)$ by o_2. First, let us consider the case where $|N(\mathcal{P})| = 4$. Then $e_x = xo_1$, $e_y = yo_2$, $e_z = zo_2$ and $e_w = wo_2$. By the choice of $\mathcal{P}$, edge xo_2 does not exist since otherwise, replacing P_x by another path v^* - x - o_2 decreases $|O(\mathcal{P})|$. For forbidding paths x - v^* - y - o_2, x - v^* - z - o_2 and x - v^* - w - o_2 from being induced subgraphs, G must contain edges xy, xz and xw (See Figure 6(a)).

Since each vertex has at least four neighbors, y has another neighbor than v^*, x and o_2. If the neighbor

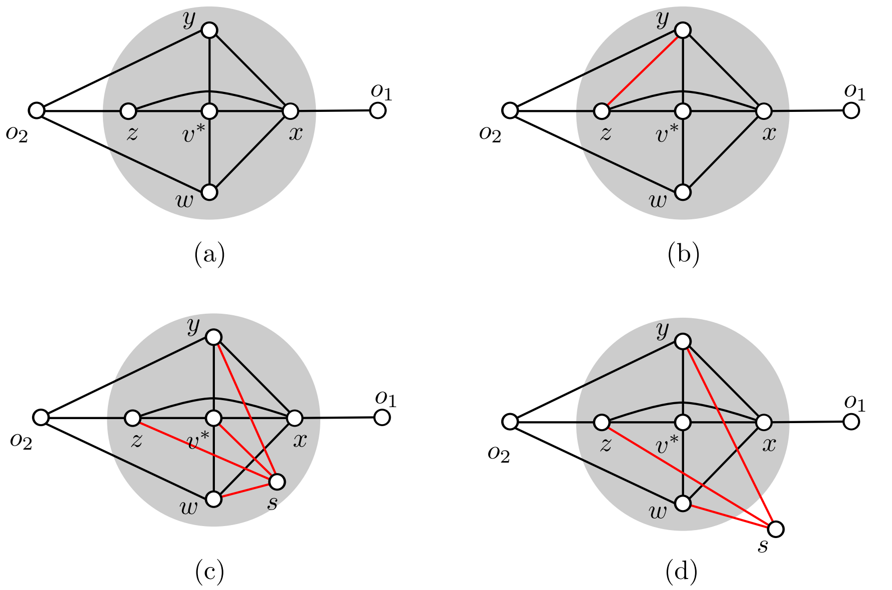

Figure 6: When $|O(\mathcal{P})| = 2$, $O(e_x) \neq O(e_y) = O(e_z) = O(e_w)$ and $|N(\mathcal{P})| = 4$

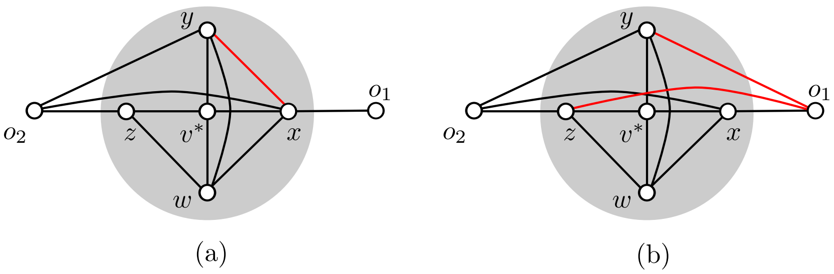

Figure 7: When $|O(\mathcal{P})| = 2$, $O(e_x) \neq O(e_y) = O(e_z) = O(e_w)$ and $|N(\mathcal{P})| = 3$

is z or w, then $G[\{v^*, x, y, z, w, o_2\}]$ contains a subgraph isomorphic to H_1 (See Figure 6(b)). Hence let the neighbor be another vertex, call s. If $s \in N(v^*)$, i.e., G contains an edge v^*s, then no edge joins s and o_2 since otherwise replacing P_x with another path v^* - s - o_2 decreases $|O(\mathcal{P})|$. In this case, for forbidding paths s - v^* - z - o_2 and s - v^* - w - o_2 from being induced subgraphs, there must be edges zs and ws (See Figure 6(c)). On the other hand, if $s \in O$, then for forbidding paths z - v^* - y - s and w - v^* - y - s from being induced subgraphs, there must be edges zs and ws (See Figure 6(d)). Therefore $G[\{v^*, x, y, z, w, o_2, s\}]$ contains H_2 in both cases.

Next, let us consider the case where $|N(\mathcal{P})| = 3$. Without loss of generality, let $e_x = xo_1$, $e_y = yo_2$, $e_z = zo_2$ and $e_w = xo_2$. Since G does not contain edge wo_2 by the minimality of $\mathcal{P}$, for forbidding paths w - v^* - x - o_2, w - v^* - y - o_2 and w - v^* - z - o_2 from being induced subgraphs, G must contain edges xw, yw and zw. If there is either xy or xz, then $G[\{v^*, x, y, z, w, o_2\}]$ contains H_1 (see Figure 7(a)). Suppose otherwise; i.e., G does not contain xy nor xz. Then, for forbidding paths y - v^* - x - o_1 and z - v^* - x - o_1 from being induced paths, there must be edges yo_1 and zo_1. In this case, $G[\{v^*, x, y, z, w, o_1, o_2\}]$ contains H_2 (see Figure 7(b)).

When $|O(\mathcal{P})| = 1$

We denote the vertex in $O(\mathcal{P})$ by o. In this case, all of $N(e_x)$, $N(e_y)$, $N(e_z)$ and $N(e_w)$ are different vertices because G is simple. Thus by Claim 1, $P_x = v^*$ - x - o, $P_y = v^*$ - y - o, $P_z = v^*$ - z - o and

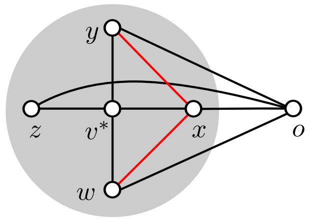

Figure 8: When $|O(\mathcal{P})| = 1$, $s, t \in \{y, z, w\}$

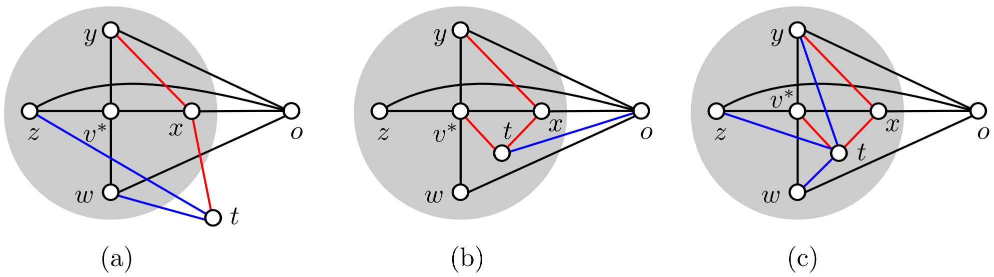

Figure 9: When $|O(\mathcal{P})| = 1$, $s = y$ and $t \notin \{y, z, w\}$

$P_w = v^*\text{-}w\text{-}o$ hold. Since G is simple and 4-edge-connected, x has at least two neighbors other than v^* and o. Call two of them s and t.

First, consider the case where we can choose both s and t from $\{y, z, w\}$ (see Figure 8). In this case, $G[\{v^*, x, y, w, o\}]$ contains W_4.

Next, consider the case where $s \in \{y, z, w\}$ (say $s = y$) and $t \notin \{y, z, w\}$. Notice that G does not contain edges xz and xw since otherwise, this case can be reduced to the case with $s, t \in \{y, z, w\}$. If $t \in O$, then G contains edges tz and tw for forbidding paths $z\text{-}v^*\text{-}x\text{-}t$ and $w\text{-}v^*\text{-}x\text{-}t$ from being induced subgraphs. Then $G[\{v^*, x, y, z, w, t, o\}]$ contains H_3 (see Figure 9(a)). Hence suppose otherwise; i.e., $t \in N(v^*)$. If G contains edge to, then $G[\{v^*, x, y, t, o\}]$ contains a W_4 (see Figure 9(b)). If G does not contain edge to, then G must contain edges ty, tz and tw for forbidding paths $t\text{-}v^*\text{-}y\text{-}o$, $t\text{-}v^*\text{-}z\text{-}o$ and $t\text{-}v^*\text{-}w\text{-}o$ from being induced subgraphs. Then $G[\{v^*, x, y, z, w, t, o\}]$ contains H_3 (see Figure 9(c)).

Next, consider the case where $s, t \notin \{y, z, w\}$. Notice that in this case, G does not contain edges xy, xz and xw since otherwise, this case can be reduced to the above cases. As in the previous case, we can observe the following facts.

- If $v \in \{s, t\}$ is in O, then G contains edges vy, vz and vw for forbidding paths $y\text{-}v^*\text{-}x\text{-}v$, $z\text{-}v^*\text{-}x\text{-}v$, $w\text{-}v^*\text{-}x\text{-}v$ from being induced subgraphs (type 1).
- If $v \in \{s, t\}$ is in $N(v^*)$, then G contains either edge vo (type 2), or all of edges vy, vz and vw (type 3).

If both s and t are type 1, then $G[\{v^*, x, y, z, w, s, t, o\}]$ contains H_4 (see Figure 10(a)). If s is type 1 and t is type 2, then $G[\{v^*, x, y, z, w, s, t, o\}]$ contains H_5 (see Figure 10(b)). If s is type 1 and t is type 3, then $G[\{v^*, x, y, z, w, s, t, o\}]$ contains H_4 (see Figure 10(c)). If both s and t are type 2, then $G[\{v^*, x, s, t, o\}]$ contains a W_4 (see Figure 10(d)). If s is type 2 and t is type 3, then $G[\{v^*, x, y, z, w, s, t, o\}]$ contains H_5 (see Figure 10(e)). If both s and t are type 3, then $G[\{v^*, x, y, z, w, s, t, o\}]$ contains H_4 (see Figure 10(f)). We can see that G contains a required subgraph in all cases.

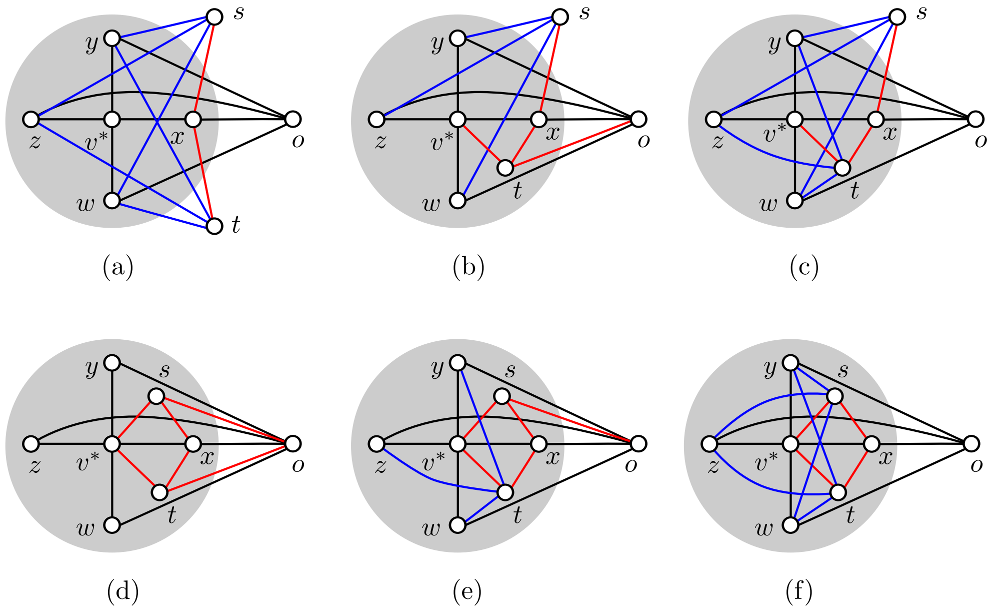

Figure 10: When $|O(\mathcal{P})| = 1$ and $s, t \notin \{y, z, w\}$

References

[1] G. Dirac, *On rigid circuit graphs*, Abhandlungen Mathematischen Seminar Universitt Hamburg, 25 (1961), pp. 71–76.

[2] F. Jaeger, N. Linial, C. Payan, and M. Tarsi, *Group connectivity of graphs - a nonhomopeneous analogue of nowhere-zero flow properties*, Journal of Combinatorial Theory Series B, 56 (1992), pp. 165–182.

[3] B. Jamison and S. Olariu, *On the semi-perfect elimination*, Advances in Applied Mathematics, 9 (1988), pp. 364–376.

[4] M. Kochol, *An equivalent version of 3-flow conjecture*, Journal of Combinatorial Theory Series B, 83 (2001), pp. 258–261.

[5] H.-J. Lai, *Group connectivity of 3-edge-connected chordal graphs*, Graphs and Combinatorics, 16 (2000), pp. 165–176.

[6] ———, *Nowhere-zero 3-flows in locally connected graphs*, Journal of Graph Theory, 42 (2003), pp. 211–219.

[7] H.-J. Lai, L. Miao, and Y. Shao, *Every line graph of a 4-edge-connected graph is $\mathbb{Z}_3$-connected*, European Journal of Combinatorics, to appear (2008).

[8] P. D. Seymour, *Nowhere-zero flows*, in Handbook of Combinatorics, R. L. Graham, M. Grötschel, and L. Lovász, eds., 1995, pp. 289–299.

[9] W. T. Tutte, *On the algebraic theory of graph colorings*, Journal of Combinatorial Theory, 1 (1966), pp. 15–50.

[10] C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs*, Marcel Dekker, Inc., 1997.