A Plane Graph Representation of Triconnected Graphs

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Abstract

Given a graph \( G = (V, E) \), a set \( S = \{s_1, s_2, \ldots, s_k\} \) of \( k \) vertices of \( V \), and \( k \) natural numbers \( n_1, n_2, \ldots, n_k \) such that \( \sum_{i=1}^{k} n_i = |V| \), the \( k \)-partition problem is to find a partition \( V_1, V_2, \ldots, V_k \) of the vertex set \( V \) such that \( |V_i| = n_i \), \( s_i \in V_i \), and \( V_i \) induces a connected subgraph of \( G \) for each \( i = 1, 2, \ldots, k \). For the tripartition problem on a triconnected graph, a naive algorithm can be designed based on a directional embedding of \( G \) in the two dimensional Euclidean space. However, for graphs of large number of vertices, implementing of this algorithm requires high precision real arithmetic to distinguish two close vertices in the plane. In this paper, we propose an algorithm to the tripartition problem by introducing a new data structure called \( \text{region graph} \), which represents some kind of combinatorial embedding of the given graph in the plane. The algorithm constructs a desired tripartition combinatorially in the sense that it does not require any geometrical computation with actual coordinates in the Euclidean space.

Key words: Algorithms, Connectivity, Plane Graphs, Graph Embedding, Data Structure, Independent Trees, Partition.

1 Introduction

Given an undirected graph \( G = (V, E) \), a set \( S = \{s_1, s_2, \ldots, s_k\} \) of \( k \) vertices of \( V \), and \( k \) natural numbers \( n_1, n_2, \ldots, n_k \) such that \( \sum_{i=1}^{k} n_i = |V| \), we wish to find a partition \( V_1, V_2, \ldots, V_k \) of the vertex set \( V \) such that \( |V_i| = n_i \), \( s_i \in V_i \), and \( V_i \) induces a connected subgraph for each \( i = 1, 2, \ldots, k \). Such a partition is called a \( k \)-partition of \( G \). The problem of partitioning a given graph into

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connected subgraphs appears in many applications such as the traffic load balancing [9], the fault-tolerant routing [11], image processing [7,14], paging systems of operation systems [20], co-design of hardware and software [5,7], and political districting [2,23].

The $k$-partition problem is shown to be NP-hard by Dyer and Frieze even if $k = 2$ [4]. Győri [8] and Lovász [13] proved independently that every $k$-connected graph has a $k$-partition, however, their results do not give a polynomial time algorithm for finding such a $k$-partition. Suzuki et al. [18] proposed a linear time algorithm to find a bipartition of a biconnected graph. For a triconnected graph, Suzuki et al. [19] proposed an $O(|V|^2)$ time algorithm to find a tripartition of the graph. Afterward, Nakano et al. [17] proposed a linear time algorithm to find a 4-partition of a 4-connected planar graph if the four vertices of $S$ lie on the boundary of one face of the graph. However, no polynomial time algorithm to find a $k$-partition of a $k$-connected graph is known for the case of $k \geq 4$. In a different version of the $k$-partition problem, where $|S| \leq k$ is allowed, we wish to find a partition $V_1, V_2, \ldots, V_k$ of $V$ such that $|V_i| = n_i$, $V_i$ induces a connected subgraph for each $i = 1, 2, \ldots, k$, and $s_i \in V_i$ for each $i = 1, 2, \ldots, |S|$. For this problem, Diwan and Kurhekar [3] proved that every triangulated plane graph admits a 6-partition, a 5-partition, and a 4-partition when $S = \emptyset$, $|S| = 1$, and $|S| = 2$, respectively.

For a graph $G = (V, E)$ and a set $S = \{s_1, s_2, \ldots, s_k\}$ of $k$ vertices of $V$, $k$-independent trees are $k$ spanning trees $T_1, T_2, \ldots, T_k$ of $G$ such that for any vertex $v \in V - S$ and any distinct numbers $i, j \in \{1, 2, \ldots, k\}$, the path from $v$ to $s_i$ in $T_i$ and the path from $v$ to $s_j$ in $T_j$ are vertex-disjoint. The problem of finding $k$-independent trees of a given graph appears in the fault-tolerant routing [12]. For this problem, Itai and Rodeh [12] designed a linear time algorithm to find 2-independent trees of a biconnected graph. Afterward, Zelavi and Itai [24] proposed an $O(|V|^2)$ time algorithm to find 3-independent trees of a triconnected graph.

A $k$-directional embedding of a graph $G = (V, E)$ is a mapping from the set of vertices in $V$ to points in $(k - 1)$-dimensional Euclidean space, such that for each vertex $v \in V$, the space is partitioned into $k$ directional regions each of which contains at least one of the neighbours of $v$. Annexstein and Berman [1] proved that every triconnected graph admits a 3-directional embedding in the Euclidean plane, and gave an algorithm for constructing 3-independent trees from a 3-directional embedding.

In this paper, we consider the tripartition problem. That is, for a given triconnected graph $G = (V, E)$ and a set $S = \{s_1, s_2, s_3\}$ of specified vertices of $V$, we wish to find a tripartition of $G$ for any natural numbers $n_1, n_2, n_3$ with $n_1 + n_2 + n_3 = |V|$. We first propose an algorithm for finding a tripartition of $G$ based on a 3-directional embedding of $G$ in the Euclidean plane. However,
for graphs of a large number of vertices, implementing such an algorithm on actual computers, where a number is represented in a limited size, may fail to terminate correctly due to numerical errors.

For the problem of bisecting a triconnected graph, Nagamochi et al. [16] showed that an implementation of their algorithm based on computing actual embedded points in the plane fails to compute a bisection correctly for instances with 1000 vertices. They overcame such a problem of embedded algorithms for the bisection problem by introducing a new way of representing the relative positions of embedded vertices, avoiding a geometrical computation with the actual coordinates in the plane. They showed that the implementation of the new algorithm can compute correctly the bisecting of graphs of a large number of vertices. However, the data structure cannot be applied to the tripartition problem.

In this paper, we construct in \(O(|V|^2)\) time a new data structure called region graph, which is a kind of combinatorial embedding of a given triconnected graph \(G = (V, E)\) so that we do not need to compute the actual coordinates of vertices embedded in the plane. We show that a tripartition and 3-independent trees of \(G\) can be obtained from its region graph in linear time.

The paper is organized as follows. Section 2 introduces some notations and propose a linear time algorithm to the tripartition problem using a directional embedding of the given graph. In Section 3, definitions and basic properties of region graphs are given. Section 4 describes how to generate a partition of a vertex set of a region graph, while an algorithm for constructing of a region graph is described in Section 5. In Section 6, we construct a consistent region graph of a given triconnected graph, based on which a tripartition and 3-independent trees of the graph are computed. Finally, we make a conclusion in Section 7.

2 Preliminaries

2.1 Notations

Let \(G\) stand for an undirected simple graph with a set \(V(G)\) of vertices and a set \(E(G)\) of edges, where we denote \(|V(G)|\) and \(|E(G)|\) by \(n\) and \(m\), respectively. For each \(v \in V\), the neighbours of \(v\) is the vertices adjacent to \(v\) in \(G\) and the set of neighbours of \(v\) is denoted by \(N(v; G)\). A graph \(G\) is called \(k\)-connected if and only if \(|V| \geq k + 1\) and the graph \(G - X\) obtained from \(G\) by removing any set \(X\) of \((k - 1)\) vertices of \(V\) remains connected.
Let $e = (u, v) \in E$ be an edge with end vertices $u$ and $v$. We denote by $G/e$ the graph obtained by contracting $u$ and $v$ into a single vertex, and by $G - e$ the graph obtained from $G$ by removing $e$. Subdividing an edge $e = (u, v)$ means that we replace $e$ by a path $P$ from $u$ to $v$, where the inner vertices of $P$ are new vertices of the graph. The graph $G'$ obtained by subdividing some edges in $G$ is called a subdivision of $G$.

2.2 Directional Embedding

In this section, we review the definition of a 3-directional embedding of a given graph in the Euclidean plane, which was proposed by Annexstein and Berman [1].

For a vertex $v$ in the Euclidean plane, let antiray-1, ray-3, antiray-2, ray-1, antiray-3, and ray-2 denote six rays from $v$, which we always draw in the this order clockwise around $v$ in the plane. Such a set of rays is called a 3-directional compass of $v$ if and only if the angle between any two adjacent rays is 60 degrees (see Fig. 1(b)).

For the 3-directional compass of a vertex $v$, the region-$i$ of $v$ is defined by the region bounded by antiray-$j$ and antiray-$k$ (not containing antiray-$i$), where $j, k \in \{1, 2, 3\} \setminus \{i\}$ ($i = 1, 2, 3$).

![Diagram](a) ![Diagram](b) ![Diagram](c)

Fig. 1. Illustration of a directional $S$-embedding of a given graph; (a) A 3-connected graph $G$; (b) A 3-directional compass of a vertex $v$; (c) A directional $S$-embedding of $G$ without all rays.

**Definition 1** Let $G = (V, E)$ be a graph, and $S = \{s_1, s_2, s_3\}$ be a set of three vertices of $V$. An embedding of $G$ in the Euclidean plane is called a directional $S$-embedding of $G$ if and only if the embedding satisfies the following conditions (see Fig. 1).

(i) For any $v \in V - S$, each of the three regions of $v$ contains a neighbour of
v. 
(ii) For any \( v \in V - S \) and \( i \in \{1, 2, 3\} \), the region-\( i \) of \( v \) contains the vertex \( s_i \).
(iii) Every antiray contains no vertices of \( V \) other than its origin, and any rays or antirays of three distinct vertices do not intersect at the same point.
(iv) For any two distinct vertices \( u, v \in V \), the same kind of rays of \( u \) and \( v \) are parallel and have the same directions.

2.3 Computing a tripartition by directional embedding

In this section, we compute a tripartition of a triconnected graph by using a directional \( S \)-embedding of the graph.

**Lemma 2** Let \( G = (V, E) \) be a graph, \( S = \{ s_1, s_2, s_3 \} \) be a set of three vertices of \( V \), and assume that \( G \) has a directional \( S \)-embedding \( \psi \). Place a 3-directional compass on a point \( p \) in the plane so that none of the antirays of the compass contains a vertex \( v \in V \) embedded in \( \psi \). Let \( V_i \subseteq V \) denote the set of all vertices in region-\( i \) of the compass for each \( i = 1, 2, 3 \). Then, for each \( i = 1, 2, 3 \), if \( V_i \neq \emptyset \), then \( s_i \in V_i \) and the subgraph of \( G \) induced by \( V_i \) is connected.

**Proof:** Assume that \( V_i \neq \emptyset \) holds for some \( i = 1, 2, 3 \). If \( s_i \notin V_i \) holds, then for any \( u \in V_i \), \( s_i \) exists outside the region-\( i \), which contradicts Definition 1(ii). Hence \( s_i \in V_i \) holds.

We show that for any \( u \in V_i \), there is a path between \( u \) and \( s_i \) in the subgraph of \( G \) induced by \( V_i \). If \( u = s_i \), then we are done. Assume that \( u \neq s_i \). Note that, for any two distinct vertices \( x, y \in V \), if the region-\( i \) of \( x \) contains \( y \), then the region-\( i \) of \( x \) contains the region-\( i \) of \( y \) properly. Therefore, by Definition 1(i), \( u \) is adjacent to either \( s_i \) or a vertex \( u' \) which region-\( i \) is contained in the region-\( i \) of \( u \) properly. Hence from Definition 1(ii), there must be a path between \( u \) and \( s_i \) in the subgraph of \( G \) induced by \( V_i \). This implies that the subgraph of \( G \) induced by \( V_i \) is connected (since \( u \) is an arbitrary vertex of \( V_i \)). \( \square \)

**Theorem 3** Let \( G = (V, E) \) be a graph, \( S = \{ s_1, s_2, s_3 \} \) be a set of three vertices of \( V \), and \( n_1, n_2, \) and \( n_3 \) be three natural numbers with \( n_1 + n_2 + n_3 = n \). Then, given a directional \( S \)-embedding of \( G \), a tripartition of \( G \) can be obtained in linear time.

**Proof:** Place a 3-directional compass on a point \( p \) in the plane so that none of the antirays of the compass contains a vertex \( v \in V \) embedded in \( \psi \). Let \( V_i \subseteq V \) denote the set of all vertices in region-\( i \) of the compass for each
\[ i = 1, 2, 3. \text{ That is, such a 3-directional compass defines an ordered partition (} V_1, V_2, V_3) \text{ of } V. \text{ From Lemma 2, if } V_i \neq \emptyset \text{ holds, then the subgraph of } G \text{ induced by } V_i \text{ is connected and } s_i \in V_i \text{ holds for all } i = 1, 2, 3. \text{ It remains to prove that, for any three natural numbers } n_1, n_2, \text{ and } n_3 \text{ with } n_1 + n_2 + n_3 = n, \text{ we can place a 3-directional compass satisfying } |V_i| = n_i \text{ for all } i = 1, 2, 3. \text{ We start by placing a compass such that all vertices of the graph are contained in its region-3, from which we get a partition } (0, 0, V) \text{ of } V. \text{ If } |V_i| < n_i, \text{ then we repeatedly move the compass to decrease } |V_3| \text{ by 1, and increase } |V_i| \text{ by 1, without changing } |V_2| = 0 \text{ until } |V_i| = n_i \text{ holds. Finally, we repeatedly move the compass to decrease } |V_3| \text{ by 1, and increase } |V_2| \text{ by 1, without changing } |V_i| \text{ until } |V_2| = n_2 \text{ holds. Obviously, the obtained partition } (V_1, V_2, V_3) \text{ be such that } |V_i| = n_i \text{ for all } i = 1, 2, 3. \]

Now, we show that, for any distinct \( i, j, k \in \{1, 2, 3\} \) such that \( V_i \neq \emptyset \) holds, we can decrease \( |V_i| \) by 1, and increase \( |V_j| \) by 1, without changing \( |V_k| \) by moving a 3-directional compass. Without loss of generality we assume that \( (i, j, k) = (1, 2, 3) \). We move the compass in the direction of its ray-1. From Definition 1(iii), it follows that no two distinct vertices are contained in any antiray of the compass and no three distinct vertices are contained in antirays of the compass. Hence when we move the compass in the direction of its ray-1, exactly one vertex \( v \in V_1 \) touches at first either antiray-3 or antiray-2 of the compass. If \( v \) touches antiray-3, then we can decrease \( |V_1| \) by 1, and increase \( |V_2| \) by moving the compass in the direction of its ray-1 slightly. If \( v \) touches antiray-2, then we move the compass in the direction of its antiray-2. In the latter case, (i) a vertex \( v' \in V_1 \) touches antiray-3, or (ii) a vertex \( u \in V_2 \) touches antiray-1. If a vertex \( v' \in V_1 \) touches antiray-3, then we can decrease \( |V_2| \) by 1, and increase \( |V_3| \) by moving the compass in the direction of its antiray-2 slightly. If a vertex \( u \in V_3 \) touches antiray-1, then we can decrease \( |V_3| \) by 1, and increase \( |V_2| \) by 1 by moving the compass so that \( V_2 \) and \( V_3 \) contain \( u \) and \( v \), respectively. \( \square \)

3 Region Graphs

The purpose of this section is to introduce a new data structure called region graph and to discuss its basic properties. Such a data structure will be the basis of our algorithm to the tripartition problem.

3.1 Definition of Region Graphs

To represent the structural configuration of a directional \( S \)-embedding of a graph \( G \) without using the actual coordinates, we first introduce a simple
closed curve which encloses the embedded vertices and the intersections by all
pairs of half lines in the 3-directional compasses. Then we draw all the line
segments (some from the curve subdivided by the half lines) as a plane graph.
Thus we introduce a plane graph $H = (V, W, F)$ with two disjoint vertex sets
$V$ and $W$ and an edge set $F$, where $V$ is used to represent the vertex set of
$G$, $W$ represents the set of all the intersections, and $F$ is the set of edges that
represent the line segments. Let $B$ denote the outer facial cycle of the boundary
of $H$. Let $F$ consist of boundary edges ($b$-edges), blue, green and red compass
edges ($c$-edges), and blue, green and red anticompass edges ($a$-edges). For each
$v \in V$, the path from $v$ which consists of the same type of edges (such as $b$-
edges) is called a radial path. The radial path of $v$ which consists of blue (resp.,
green and red) $c$-edges is denoted by $P^+_1(v)$ (resp., $P^+_2(v)$ and $P^+_3(v)$), and the
radial path of $v$ which consists of blue (resp., green and red) $a$-edges is denoted
by $P^-_1(v)$ (resp., $P^-_2(v)$ and $P^-_3(v)$) (see Fig. 2). A “region graph” is define
as the following plane graph $H$, which exploits some structural properties of
directinal $S$-embeddings, but is not necessarily constructed from a particular
directinal $S$-embedding.

**Definition 4** The above plane graph $H = (V, W, F)$ with a set $S = \{s_1, s_2, s_3\} \subseteq V$ is called a region graph if the following ten conditions hold:

1. The boundary $B$ is a cycle which consists of all $b$-edges.
2. The degree of each vertex in $V$ is 6, the degree of each vertex in $W - W(B)$
is 4 and the degree of each vertex in $W(B)$ is 3.
3. All vertices of $H$ are contained in the interior of $B$ and on $B$.
4. For each vertex $v \in V$, every radial path of $v$ ends with a vertex with
$B$ and exactly six radial paths $P^+_1(v)$, $P^-_1(v)$, $P^+_2(v)$, $P^-_2(v)$, $P^+_3(v)$ and
$P^-_3(v)$ exist clockwise around $v$ (see Fig. 2).
5. No two radial paths with the same color share a vertex.
6. For each vertex $v \in V$, no two radial paths of $v$ meet at other vertices.
7. For any two vertices $u, v \in V$, one radial path of $u$ and one radial path
of $v$ share at most one vertex.
8. For each vertex $v \in V$, no vertex in $V - \{v\}$ is contained in any radial
path of $v$.
9. For any distinct vertices $u, v \in V$, three paths \{${P^-_1(u), P^-_2(u), P^-_3(u)}$
and three paths \{${P^-_1(v), P^-_2(v), P^-_3(v)}$\} have exactly one common vertex.
10. For each vertex $v \in V - \{s_i\}$, $P^-_i(v)$ intersects with either $P^-_j(s_i)$ or
$P^-_k(s_i)$ for any distinct $i, j, k \in \{1, 2, 3\}$.

Note that, there exist region graphs which cannot be derived from any direc-
tional $S$-embedding (see Fig. 3(a) for an example).

Consider a region graph $H = (V, W, F)$ with boundary $B$. For each vertex
$v \in V$ and $i = 1, 2, 3$, let $w^i_v$ denote the common vertex of $P^+_i(v)$ and $B$,
and $x^i_v$ denote the common vertex of $P^-_i(v)$ and $B$ (see Fig. 2). For any
Fig. 2. Illustration of a region graph; (a) Radial paths of a vertex $v \in V$ in a region graph $H = (V, W, F)$, where solid lines indicate $c$-edge paths, dashed lines $a$-edge paths, and the largest circle the boundary $B$ of $H$; (b) A region graph of the graph given in Fig. 1(a).

Fig. 3. (a) A region graph which cannot be derived from any directional $S$-embedding. (b) Illustration of Lemma 6(iii)-(iv). $V - \{s_i\} \subseteq AR_i(s_i),$ $i = 1, 2, 3.$ All vertices in $V - S$ are contained in the grey region.

vertices $x, y \in W(B)$, let $B[x, y]$ denote the subpath of $B$ obtained by running clockwisely on $B$ from $x$ to $y.$ For each $(i, j, k) \in \{(1, 3, 2), (2, 1, 3), (3, 2, 1)\}$, let $B_i$ denote $B[x^i_{s_i}, x^i_{s_j}]$ (see Fig. 3(b)).

Let $W^-$ denote the set of vertices shared by two $a$-edge radial paths of two distinct vertices, or by an $a$-edge radial path and the boundary $B.$ Let $W^+$ denote the set $W - W^-;$ i.e., $W^+$ consists of the set of vertices shared by two
c-edge radial paths, by an a-edge radial path and a c-edge radial path, or by a c-edge radial path and the boundary $B$.

For a vertex $v \in V$ and $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, the region bounded by $P_{j}^{+}(v)$, $P_{k}^{-}(v)$ and $B[u_v, w_v]$ is called the region-$i$ of $v$, where we consider that the region-$i$ of $v$ contains its boundary. Let $R_i(v) \subseteq V \cup W$ denote the set of all vertices contained in the region-$i$ of $v$ (see Fig. 4(a)). Similarly, the region bounded by $P_{j}^{-}(v)$, $P_{k}^{-}(v)$, and $B[x_v^j, x_v^k]$ is called the antiregion-$i$ of $v$, where we consider that the antiregion-$i$ of $v$ contains its boundary. Let $AR_i(v) \subseteq V \cup W$ denote the set of all vertices contained in the antiregion-$i$ of $v$ (see Fig. 4(b)).

![Diagram](image)

**Fig. 4.** (a) Regions of a vertex $v \in V$; (b) Antiregions of a vertex $v \in V$.

### 3.2 Basic Properties of Region Graphs

For each vertex $v \in V$, we let $P_i(v)$, $i = 1, 2, 3$, denote the path which consists of $P_i^+(v)$ and $P_i^-(v)$.

**Lemma 5** Let $H = (V, W, F)$ be a region graph. Then, for any two distinct vertices $u, v \in V$ and any distinct $i, j \in \{1, 2, 3\}$, $P_i(u)$ and $P_j(v)$ cannot share more than one vertex.

**Proof:** Assume for the contrary that $P_i(u)$ and $P_j(v)$ share two vertices. From Condition 7, it follows that $P_i^+(u)$ and $P_j^+(v)$ (resp., $P_i^-(u)$ and $P_j^-(v)$) cannot share two vertices. Therefore, without loss of generality we can assume that $P_i^+(u)$ intersects both of $P_j^+(v)$ and $P_j^-(v)$. Hence, by Condition 4, $P_i^+(u)$ and $P_i(v)$ share a vertex, which contradicts Condition 5. Consequently, $P_i(u)$ and $P_j(v)$ cannot share more than one vertex. □
Fig. 5. Illustration of Lemma 6: (a) \( s_2 \in AR_2(s_1) \); (b) \( s_2 \in AR_3(s_1) \); (c) \( v \in AR_2(s_1) \) for \( v \in V - S \).

**Lemma 6** Let \( H = (V, W, F) \) be a region graph with boundary \( B \). Then \( H \) satisfies the following conditions.

1. \( P_i^-(s_j) \) intersects with \( P_j^-(s_i) \) for any \( i, j \in \{1, 2, 3\} \).
2. \( s_1, s_2 \in AR_k(s_k) \) for any distinct \( i, j, k \in \{1, 2, 3\} \).
3. For each \( i = 1, 2, 3 \), it holds \( V - \{s_i\} \subseteq AR_i(s_i) \).
4. \( B_1, B_2 \) and \( B_3 \) share no vertices, and appear in this order along \( B \) when we traverse \( B \) in the clockwise order.
5. For each vertex \( v \in V \) and \( i = 1, 2, 3 \), it hold \( x_i^v \in V(B_i) \) and \( w_i^v \notin V(B_i) \).

**Proof:** (i) By Condition 10, \( P_i^-(s_2) \) intersects with either \( P_2^-(s_1) \) or \( P_3^-(s_1) \), and \( P_2^-(s_1) \) intersects with either \( P_3^-(s_2) \) or \( P_1^-(s_2) \). Then, it follows from Condition 9 that \( P_i^-(s_2) \) intersects with \( P_2^-(s_1) \). Analogously, we can show that \( P_2^-(s_3) \) intersects with \( P_3^-(s_2) \) and that \( P_3^-(s_1) \) intersects with \( P_1^-(s_3) \).

(ii) Assume without loss of generality that \( k = 1 \). We show that \( s_2 \in AR_1(s_1) \) (\( s_3 \in AR_1(s_1) \) can be treated analogously). First, assume that \( s_2 \in AR_2(s_1) \) (see Fig. 5(a)). From (i), \( P_1^-(s_2) \) intersects with \( P_2^-(s_1) \). This implies that \( P_1^-(s_2) \) intersects with either \( P_1^-(s_1) \) or \( P_1^+(s_1) \), which contradicts Condition 5. Hence \( s_2 \notin AR_2(s_1) \) holds. Analogously, we can show that \( s_3 \notin AR_3(s_1) \) and \( s_3 \notin AR_3(s_2) \) hold.

Next, assume that \( s_2 \in AR_3(s_1) \) (see Fig. 5(b)). From (i), we see that \( P_1^-(s_2) \) intersects with \( P_3^-(s_1) \). Hence, from Condition 9 with \( u = s_1 \) and \( v = s_2 \), we have \( AR_3(s_1) \cup AR_3(s_2) \cup (AR_1(s_1) \cap AR_2(s_2)) = V \cup W \). Since we have proved that \( s_3 \notin AR_3(s_1) \) and \( s_3 \notin AR_3(s_2) \), it holds \( s_3 \in AR_1(s_1) \cap AR_2(s_2) \). From (i), we also see that \( P_3^-(s_1) \) intersects with \( P_3^-(s_2) \). This implies that \( P_3^-(s_3) \) intersects with either \( P_1^-(s_2) \) or \( P_1^+(s_2) \), which contradicts Condition 5. Hence \( s_2 \in AR_1(s_1) \) must hold (since \( s_2 \notin AR_2(s_1) \) and \( s_2 \notin AR_3(s_1) \)).
(iii) Assume without loss of generality that $i = 1$. For any vertex $v \in V - S$, we first show that $v \notin AR_2(s_1)$. Assume for the contrary that $v \in AR_2(s_1)$ holds (see Fig. 5(c)). From Conditions 9 and 10, it follows that $P_i^-(v)$ intersects with $P_1^-(s_1)$. Also, Condition 9 implies that $V(P_2^-(v)) \subseteq AR_2(s_1)$ holds. From (i), we see that $P_i^-(s_2)$ intersects with $P_2^-(s_1)$, where $s_2 \in AR_1(s_1)$ holds by (ii). Hence, it follows from Condition 9 that neither $P_i^-(s_2)$ nor $P_3^-(s_2)$ has a vertex in the antiregion-2 of $s_1$. This implies that $P_2^-(v)$ cannot intersect with $P_i^-(s_2)$ or $P_3^-(s_2)$, which contradicts Condition 10. Hence $v \notin AR_2(s_1)$ holds. Similarly, we can show that $v \notin AR_3(s_1)$. Consequently, $V \setminus \{s_1\} \subseteq AR_1(s_1) \setminus AR_1(s_1)$ by (ii).

(iv) From (i) and (ii), we observe that $x^{s_1}_v, x^{s_2}_v, x^{s_3}_v, x^{s_4}_v, x^{s_5}_v, x^{s_6}_v, x^{s_7}_v, x^{s_8}_v$ appear in this order along $B$ when we traverse $B$ in the clockwise order (see Fig. 3(b)). Hence $B_1, B_2$ and $B_3$ appear in this order along $B$ when we traverse $B$ in the clockwise order and hence they share no vertices.

(v) Assume without loss of generality that $i = 1$. From the definition of $B_i$, it holds $x^{s_1}_v, x^{s_2}_v, x^{s_3}_v, x^{s_4}_v, x^{s_5}_v, x^{s_6}_v, x^{s_7}_v, x^{s_8}_v \in V(B_1)$. Note that, it follows from (iii) that $v \in AR_i(s_1) \cap AR_2(s_2) \cap AR_3(s_3)$ for any vertex $v \in V - S$. We first show that $x^{s_1}_v \in V(B_1)$. From Conditions 10, $P_i^-(v)$ intersects with either $P_2^-(s_1)$ or $P_3^-(s_1)$. Therefore, $x^{s_1}_v \in V(B_1)$ holds, by Conditions 5 and 9.

Now, we show that $u_i^v \notin V(B_1)$. Assume for the contrary that $u_i^v \in V(B_1)$. Since $P_i^-(v), P_2^-(v)$, and $P_i^-(v)$ appear around $v$ in this order clockwise, it follows that $x^{s_2}_v \in V(B_1)$. This contradicts $x^{s_2}_v \in V(B_2)$. Since $B_1$ and $B_2$ share no vertices. □

Lemma 7 Let $H = (V, W, F)$ be a region graph with boundary $B$. Then, for any $i \in \{1, 2, 3\}$, $j, k \in \{1, 2, 3\} \setminus \{i\}$, and $v \in V - \{s_i\}$, $P_i^-(v)$ reaches some vertex $w \in W(B) - AR_i(s_i)$, and both $P_j^-(v)$ and $P_k^-(v)$ cannot go out from the antiregion-$i$ of $s_i$ (hence $V(P_j^-(v)) \cup V(P_k^-(v)) \subseteq AR_i(s_i)$).

Proof: Without loss of generality we prove the lemma for $i = 1$. Note that, from Lemma 6(ii), it holds $V \setminus \{s_i\} \subseteq AR_i(s_i)$. By Condition 10, $P_i^-(v)$ intersects with either $P_2^-(s_1)$ or $P_3^-(s_1)$. Therefore, Conditions 4 and 9 imply that $P_i^-(v)$ reaches some vertex $w \in W(B) - AR_i(s_1)$. On the other hand, by Condition 9, none of $P_2^-(v)$ and $P_3^-(v)$ intersects with any a-edge path of $s_1$ (since $P_i^-(v)$ intersects with one of $P_2^-(s_1)$ and $P_3^-(s_1)$). Hence $V(P_2^-(v)) \cup V(P_3^-(v)) \subseteq AR_i(s_1)$. □

Lemma 8 Let $H = (V, W, F)$ be a region graph with boundary $B$, and let $w \in W^-$ be a common vertex of two $a$-edge paths. Then $w \in AR_i(s_1) \cap AR_2(s_2) \cap AR_3(s_3)$ holds.
**Proof:** Let \( w \) be a common vertex of an a-edge path of \( u \) and an a-edge path of \( v \) for two distinct vertices \( u, v \in V \). Assume for the contrary that \( w \notin AR_1(s_1) \cap AR_2(s_2) \cap AR_3(s_3) \) holds. Without loss of generality we can assume that \( w \in AR_2(s_1) \). By Lemma 6(iii), it holds \( u, v \in AR_1(s_1) \). Therefore, by Lemma 7, two paths \( P_1^{-}(u) \) and \( P_1^{-}(v) \) go out from the antiregion-1 of \( s_1 \), and four paths \( P_2^{-}(u), P_3^{-}(u), P_2^{-}(v), P_3^{-}(v) \) cannot go out from the antiregion-1 of \( s_1 \). Hence \( w \) is a common vertex of \( P_1^{-}(u) \) and \( P_1^{-}(v) \), which contradicts Condition 5. Consequently, \( w \in AR_1(s_1) \cap AR_2(s_2) \cap AR_3(s_3) \) holds. □

**Lemma 9** Let \( H = (V, W, F) \) be a region graph with boundary \( B \). Then, for any two vertices \( u, v \in V \) and \( i \in \{1, 2, 3\} \), the following statements are equivalent.

(i) \( P_i^{-}(u) \) intersects an a-edge radial path of \( v \);
(ii) \( u \in AR_i(v) \) holds; and
(iii) \( v \in R_i(u) \) holds.

**Proof:** Without loss of generality we prove the lemma for \( i = 1 \).

(i) \( \Rightarrow \) (ii) Assume that \( P_1^{-}(u) \) intersects an a-edge radial path of \( v \). First, assume that \( u \in AR_2(v) \). Since \( u \) is in the region bounded by \( P_1^{-}(v), P_3^{-}(v) \) and \( B[x_u^{1}, x_v^{1}] \), it follows from Conditions 5 and 9 that \( P_1^{-}(u) \) intersects with \( P_3^{-}(v) \) and thereby \( x_u^{1} \in V(B[x_u^{2}, x_v^{2}]) \) (see Fig. 6(a)). On the other hand, it holds \( x_u^{1}, x_v^{1} \in B_i \) by Lemma 6(v). This implies that \( x_u^{2} \in B_i \) or \( x_v^{2} \in B_i \) holds, which contradicts Lemma 6(iv)-(v). Hence \( u \notin AR_2(v) \). Analogously, we can show that \( u \notin AR_3(v) \) holds. Consequently, \( u \in AR_1(v) \) holds.

(ii) \( \Rightarrow \) (iii) Assume that \( u \in AR_1(v) \) holds. By Lemma 6(iv)-(v), \( P_1^{-}(u) \) intersects with either \( P_2^{-}(v) \) or \( P_3^{-}(v) \). Without loss of generality we can assume that \( P_1^{-}(u) \) intersects with \( P_2^{-}(v) \). Now, if \( v \notin R_1(u) \), then it follows from Condition 4 that \( P_2^{+}(u) \) intersects with \( P_3^{-}(v) \) (see Fig. 6(b)), which contradicts

![Fig. 6. Illustration of Lemma 9: (a) \( u \in AR_2(v) \); (b) \( u \in AR_1(v) \) and \( v \notin R_1(u) \); (c) \( v \in AR_3(u) \cap R_1(u) \).](image-url)
Condition 5. Hence \( v \in R_i(u) \) holds.

(iii) \( \Rightarrow \) (i) Assume that \( v \in R_i(u) \) holds. Then either \( v \in AR_2(u) \cap R_i(u) \) or \( v \in AR_3(u) \cap R_i(u) \). If \( v \in AR_2(u) \cap R_i(u) \), then by Lemma 6(iv)-(v) and Condition 5, \( P^+_2(v) \) intersects with \( P^-_3(u) \). If \( v \in AR_3(u) \cap R_i(u) \) holds, then \( P^-_3(v) \) intersects with \( P^-_1(u) \) (see Fig. 6(c)). Hence if \( v \in R_i(u) \) holds, then \( P^-_1(u) \) intersects an a-edge radial path of \( v \). \( \Box \)

**Lemma 10** Let \( H = (V, W, F) \) be a region graph with boundary \( B \). Then, for each two vertices \( u, v \in V \) and \( i, j \in \{1, 2, 3\} \), the following statements are equivalent.

(i) \( P^-_i(u) \) and \( P^-_j(v) \) share a vertex (such \( i \) and \( j \) are unique by Condition 9);

(ii) \( v \in AR_j(u) \) and \( u \in AR_i(v) \) hold; and

(iii) \( v \in R_i(u) \) and \( u \in R_j(v) \) hold.

**Proof:** The proof follows directly from Lemma 9. \( \Box \)

**Lemma 11** Let \( H = (V, W, F) \) be a region graph. Then, for a vertex \( u \in V \) and \( i \in \{1, 2, 3\} \), the following (i) and (ii) hold.

(i) \( R_i(v) \subset R_i(u) \) holds for all vertices \( v \in R_i(u) - \{u\} \).

(ii) \( AR_i(v) \subset AR_i(u) \) holds for all vertices \( v \in AR_i(u) - \{u\} \).

**Proof:** Without loss of generality we prove the lemma for \( i = 1 \).

(i) Let \( v \neq u \) be an arbitrary vertex in \( R_1(u) \). By Lemma 9, \( u \in AR_1(v) \) holds. Therefore, Conditions 4 and 5 imply that \( R_1(v) \subset R_i(u) \) holds (since \( u \notin R_1(v) \)).

(ii) Let \( v \neq u \) be an arbitrary vertex in \( AR_1(u) \). By Lemma 9 and Condition 5, \( P^-_1(v) \) intersects with either \( P^-_2(u) \) or \( P^-_3(u) \). Condition 9 also imply that \( P^-_2(v) \) and \( P^-_3(v) \) cannot go out of antiregion-1 of \( u \), that is, \( V(P^-_2(v)) \cup V(P^-_3(v)) \subset AR_1(u) \). Therefore, Condition 4 implies that \( AR_1(v) \subset AR_1(u) \) (since \( u \notin AR_1(v) \)). \( \Box \)

4 Partitioning via Region Graphs

In this section, given natural numbers \( n_1, n_2 \) and \( n_3 \) with \( n_1 + n_2 + n_3 = |V| \), we consider how to define a partition \( V_1, V_2, \) and \( V_3 \) of \( V \) in a region graph \( H = (V, W, F) \) such that, for each \( i = 1, 2, 3 \), it hold \( |V_i| = n_i \) and \( s_i \in V_i \).
We begin by some basic definitions. Consider a region graph \( H = (V, W, F) \) with boundary \( B \). A \textit{wall-path} is a directed subpath of an a-edge radial path or \( B \) whose endvertices belong to \( V \cup W^- \) and whose other vertices belong to \( W^+ \). A wall-path in an a-edge path has the same direction as the a-edge path, and a wall-path in \( B \) is directed clockwise along \( B \).

A cycle consisting of wall-paths is called a \textit{cell} if its interior contains no a-edges. Two cells \( C \) and \( C' \) are \textit{adjacent} if they share a vertex. We traverse the boundary of a cell \( C \) in the clockwise order. If a wall-path has a forward (reverse) direction, we call the wall-path a forward (reverse) wall-path in \( C \). If a cell \( C \) has a forward wall-path in a blue a-edge path, a green a-edge path, a red a-edge path, or the boundary \( B \), then we write them as \( WP_1^c, WP_2^c, WP_3^c, \) and \( WP_B^c \), respectively. Similarly, if \( C \) has a reverse wall-path in a blue a-edge path, a green a-edge path, or a red a-edge path, then we may write them as \( WP_1^- \), \( WP_2^- \), and \( WP_3^- \), respectively (see Fig. 7(a)).

**Fig. 7.** (a) Illustration of wall-paths and cells in a region graph. A cell \( C \) is bounded by the wall-path of \( P_1^c(c) \) from \( u \) to \( x_3^c \), the wall-path of \( P_1^c(s_3) \) from \( u \) to \( v \), the wall-path of \( P_3^c(s_1) \) from \( v \) to \( x_3^c \), and the wall-path of \( B \) from \( x_3^c \) to \( x_3^{s_1} \), (b) Illustration for Lemma 15. A cell \( C \) has wall-paths in \( P_1^c(u) \) and \( P_3^c(v) \).

**Lemma 12** Let \( H = (V, W, F) \) be a region graph, and let \( C \) be a cell in \( H \). Then, for each \( v \in V \) and some \( i \in \{1, 2, 3\} \), it holds \( V(C) \subseteq AR_i(v) \).

**Proof:** Assume for the contrary that \( V(C) \subseteq AR_i(v) \) does not hold for all \( i = 1, 2, 3 \). Then the interior of \( C \) contains an a-edge from some a-edge radial path of \( v \), which contradicts the definition of a cell. \( \square \)

The following corollary is a direct consequence of the previous lemma.

**Corollary 13** For a region graph \( H = (V, W, F) \), each cell \( C \) of \( H \) defines an ordered partition \((V_1, V_2, V_3)\) of \( V \) such that \( V_i = \{v \in V \mid V(C) \subseteq AR_i(v)\} \), \( i = 1, 2, 3 \).
Theorem 14 Let $H = (V, W, F)$ be a region graph, $C$ be a cell of $H$, and $(V_1, V_2, V_3)$ be the ordered partition of $V$ defined by $C$. Then, for any $i \in \{1, 2, 3\}$ with $V_i \neq \emptyset$, there is a cell $C'$ adjacent to $C$ such that $C'$ defines an ordered partition $(V'_1, V'_2, V'_3)$ of $V$ which satisfies one of the following conditions.

(i) For some $v \in V$ and $j \in \{1, 2, 3\} - \{i\}$, it hold $V'_i = V_i - \{v\}$ and $V'_j = V_j \cup \{v\}$; and

(ii) For some $u, v \in V$ and $j, k \in \{1, 2, 3\} - \{i\}$, it hold $V'_i = V_i - \{v\}$, $V'_j = V_j \cup \{u\}$, and $V'_k = (V_k \cup \{v\}) - \{u\}$. □

Before proving Theorem 14, we discuss the following few lemmas.

Lemma 15 Let $H = (V, W, F)$ be a region graph with boundary $B$, and let $C$ be a cell of $H$. Then $C$ satisfies the following (i), (ii), and (iii).

(i) If $WP_i$ is the wall-path followed by $WP_j$ along $C$ in the clockwise order (or $WP_i$ is the wall-path followed by $WP_j$ along $C$), then $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$.

(ii) If $WP_i$ is the wall-path followed by $WP_j$ along $C$ in the clockwise order (or $WP_i$ is the wall-path followed by $WP_j$ along $C$), then $(i, j) \in \{(1, 3), (2, 1), (3, 2)\}$.

(iii) If $C$ has $WP_{i_1}$, then the previous wall-path of $WP_{i_1}$ along $C$ is $WP_i$ and the wall-path next to $WP_{i_1}$ along $C$ is $WP_j$ for some $i, j \in \{1, 2, 3\}$.

Proof: (i) Assume without loss of generality that $WP_i$ is the wall-path followed by $WP_j$ along $C$. By Condition 5, $j \neq 1$. Assume that $j = 3$. Then $C$ has a wall-path in $P_3^+(u)$ and $P_3^-(v)$ for some $u, v \in V$, and hence $v \in AR_2(u)$ holds (see Fig. 7(b)). This contradicts Lemma 10 with $i = 1$ and $j = 3$. Hence $j = 2$.

(ii) Analogous to (i).

(iii) Since all wall-paths in a-edge radial paths have a direction toward the boundary $B$, the previous wall-path of $WP_{i_1}$ is $WP_i$ and the next wall path of $WP_{i_1}$ is $WP_j$ for some $i, j \in \{1, 2, 3\}$. □

Lemma 16 Let $H = (V, W, F)$ be a region graph with boundary $B$, and let $C$ be a cell which contains no wall-paths in $B$. Then, for each $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 2, 1)\}$, $C$ satisfies the following (i) and (ii).

(i) $C$ has $WP_i$ or a common vertex of $WP_j$ and $WP_k$; and

(ii) $C$ has $WP_j$ or a common vertex of $WP_i$ and $WP_k$. 

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Proof: Assume without loss of generality that \((i, j, k) = (1, 2, 3)\).

(i) Assume that \(C\) does not have \(WP_1^I\). We first show that \(C\) has \(WP_2^R\). Assume for the contrary that \(C\) does not have \(WP_2^R\). Then by Lemma 15, the next wall-path of \(WP_3^R\) is either \(WP_1^I\) or \(WP_2^R\). Since \(C\) has neither \(WP_1^I\) nor \(WP_2^R\), this, however, \(C\) does not have \(WP_3^R\). Analogously, we can prove that \(C\) has no wall-paths, contradicts the definition of a cell. Hence \(C\) has \(WP_2^R\). Note that, by Lemma 15, the next wall-path of \(WP_2^R\) is \(WP_3^R\) since \(C\) does not have \(WP_1^I\). Hence \(C\) has a common vertex of \(WP_2^R\) and \(WP_3^R\). This completes the proof of (i). The proof of (ii) is analogous. 

Lemma 17 Let \(H = (V, W, F)\) be a region graph with boundary \(B\), \(C\) be a cell which has a wall-path of \(B\), and \((V_1, V_2, V_3)\) be the ordered partition of \(V\) defined by \(B\). Then, for each \((i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\), if \(V_i \neq \emptyset\) holds, then \(C\) satisfies the following (i) and (ii).

(i) \(C\) has \(WP_j^I\) or a common vertex of \(WP_k^R\) and \(WP_1^I\); and

(ii) \(C\) has \(WP_k^R\) or a common vertex of \(WP_1^I\) and \(WP_j^I\).

Proof: Assume without loss of generality that \((i, j, k) = (1, 2, 3)\).

(i) Note that, from Lemma 6(iv), \(B\) can be partitioned into the three sub-paths, \(B[x_{s_1}, x_{s_2}]\), \(B[x_{s_2}, x_{s_3}]\), and \(B[x_{s_3}, x_{s_1}] = B_3\) (see Fig. 3(b)). Thus, by assumptions, \(C\) contains a wall-path of one of such partitions.

(a) \(C\) has a wall-path of \(B[x_{s_1}, x_{s_2}]\). Then by Condition 4, \(C\) is not in the antiregion-1 of \(s_1\). By Lemma 6(iii), \(V - \{s_1\} \subseteq AR_1(s_1)\). Moreover, for all \(v \in V - \{s_1\}\), it holds \(AR_1(v) \subseteq AR_1(s_1)\), by Lemma 11(iii). Therefore, there is no vertex \(v \in V\) such that \(V(C) \subseteq AR_1(v)\). This implies that \(V_i = \emptyset\), which contradicts assumptions of the lemma.

(b) \(C\) has a wall-path of \(B[x_{s_2}, x_{s_3}]\). Note that, for each vertex \(v \in V\), it holds \(x_v^2 \in V(B_2)\), by Lemma 6(v) \((B_2 = B[x_{s_1}, x_{s_3}]\). Therefore, no a-edge radial path reaches any vertex in \(V(B[x_{s_2}, x_{s_3}])\). Hence \(C\) has \(WP_2^R\).

(c) \(C\) has a wall-path of \(B_3\). By Lemma 6(v), for each vertex \(v \in V\), it holds \(x_v^3 \in V(B_3)\). Hence \(C\) has \(WP_3^R\). On the other hand, by Lemma 15, the next wall-path of \(WP_3^R\) is either \(WP_1^I\) or \(WP_2^R\). Hence \(C\) has \(WP_2^R\) or a common vertex of \(WP_3^R\) and \(WP_1^I\).

This completes the proof of (i). (ii) can be proved similarly. 

Proof of Theorem 14. Assume without loss of generality that \((i, j, k) = (1, 2, 3)\). Note that, if \(C\) has \(WP_3^R\), then there is a cell \(C'\) which satisfies (i). Also, if \(C\) has a common vertex of \(WP_1^I\) and \(WP_2^R\), then there is a cell \(C'\) which satisfies (ii). Therefore, there is a cell \(C'\) which satisfies (i) or (ii) in the
case where \( C \) has no wall-path in \( B \), by Lemma 16(ii). Also, there is a cell \( C' \) which satisfies (i) or (ii) in the case where \( C \) has a wall-path in \( B \) and \( V_1 \neq \emptyset \), by Lemma 17(ii). \( \square \)

**Lemma 18** Let \( H = (V, W, F) \) be a region graph, \( C \) be a cell of \( H \), and \((V_1, V_2, V_3)\) be the ordered partition of \( V \) defined by \( C \). Then, for each \( i \in \{1, 2, 3\} \), \( v \in V_i \), and \( u \in R_i(v) \), it holds \( u \in V_i \). In particular, if \( V_i \neq \emptyset \) then \( s_i \in V_i \).

**Proof:** Without loss of generality, we prove the lemma for \( i = 1 \). By Corollary 13, \( V(C) \subseteq AR_1(v) \) holds. On the other hand, Lemma 9 implies that \( v \in AR_1(u) \) holds (since \( u \in R_1(v) \)). Therefore, \( AR_1(v) \subseteq AR_1(u) \) holds, by Lemma 11(ii) and hence \( V(C) \subseteq AR_1(u) \) holds. This implies that \( u \in V_1 \) by Corollary 13.

Assume that \( s_1 \notin V_1 \) to derive a contradiction. Then \( V(C) \) is not contained in \( AR_1(s_1) \) by Corollary 13. By Lemma 6(iii), \( v \in AR_1(s_1) \) for each vertex \( v \in V - \{s_1\} \) and hence \( AR_1(v) \subseteq AR_1(s_1) \) by Lemma 11(ii). Consequently, \( V_1 = \emptyset \) by Corollary 13, which contradicts assumptions of the lemma. \( \square \)

**Theorem 19** Let \( H = (V, W, F) \) be a region graph with boundary \( B \). For any natural numbers \( n_1, n_2 \) and \( n_3 \) with \( n_1 + n_2 + n_3 = |V| \), there is a cell \( C \) of \( H \) which defines an ordered partition \((V_1, V_2, V_3)\) of \( V \) such that, for each \( i = 1, 2, 3 \), it holds \( |V_i| = n_i \) and \( s_i \in V_i \) (if \( V_i \neq \emptyset \)).

**Proof:** By Lemma 18, for any cell \( C \) which defines an ordered partition \((V_1, V_2, V_3)\), if \( V_i \neq \emptyset \), then \( s_i \in V_i \).

To find a desired ordered partition of \( V \), we first find a cell of \( H \) that defines an ordered pair \((\emptyset, \emptyset, V)\) of \( V \). Lemma 6 (iii) implies that, for each \( v \in V - \{s_1, s_2\} \), it holds \( v \in AR_1(s_1) \cap AR_2(s_2) \). On the other hand, \( V(P_1^-(v)) \subseteq AR_2(s_2) \) and \( V(P_2^-(v)) \subseteq AR_1(s_1) \) hold by Lemma 7. Therefore, there is a cell \( C_0 \) such that \( V(C_0) \subseteq AR_2(s_1) \cap AR_1(s_2) \) holds and \( C_0 \) consists of wall-paths in \( P_2^- \), \( P_1^- \), and \( B \). This implies that, for each \( v \in V \), \( V(C_0) \subseteq AR_3(v) \) holds. Hence \( C_0 \) defines an ordered partition \((\emptyset, \emptyset, V)\) of \( V \).

Now, we find a cell of \( H \) which defines an ordered partition \((V_1, \emptyset, V_3)\) of \( V \) such that \( |V_1| = n_1 \). If \( n_1 = 0 \), then we are done. Assume that \( n_1 \neq 0 \). By repeatedly using Theorem 14 with \( i = 3 \) and \( j = 1 \), we find a cell which defines an ordered partition \((V_1, \emptyset, V - V_1)\) of \( V \) with \( |V_1| = n_1 \). Similarly, by repeatedly applying Theorem 14 with \( i = 3 \) and \( j = 2 \), we find a cell which defines an ordered partition \((V_1, V_2, V_3)\) of \( V \) with \( |V_1| = n_1 \) and \( |V_2| = n_2 \), which completes the proof. \( \square \)
Lemma 20 Let $H = (V, W, F)$ be a region graph with boundary $B$. For any natural numbers $n_1, n_2$ and $n_3$ with $n_1 + n_2 + n_3 = |V|$, there is exactly one cell $C$ which defines an ordered partition $(V_1, V_2, V_3)$ of $V$ such that $|V_i| = n_i$ for all $i = 1, 2, 3$.

Proof: Let $C$ and $C'$ be two distinct cells of $H$ defining ordered partitions $(V_1, V_2, V_3)$ and $(V_1', V_2', V_3')$ of $V$, respectively, such that $|V_i| = |V_i'|$ for all $i = 1, 2, 3$. Without loss of generality we can assume that one of the following two cases holds.

Case 1. There are two distinct vertices $u, v \in V$ such that $u \in V_1 \cap V_2'$ and $v \in V_2 \cap V_1'$. By Corollary 13, $V(C) \subseteq AR_1(u) \cap AR_2(v)$ and $V(C') \subseteq AR_2(u) \cap AR_1(v)$. Now, if $v \in AR_2(u)$, then $AR_2(v) \subseteq AR_2(u)$ holds, by Lemma 11, and hence $AR_1(u) \cap AR_2(v) = \emptyset$, a contradiction. Therefore, $v \notin AR_2(u)$ holds.

Analogously, we can show that $v \notin AR_1(u)$ and $u \notin AR_2(v)$ hold. Hence $v \in AR_3(u)$ and $u \in AR_3(v)$ hold, contradicting to Lemma 10 and Condition 5. Consequently, Case 1 cannot occur.

Case 2. There are three distinct vertices $u, v, x \in V$ such that $u \in V_1 \cap V_2'$, $v \in V_2 \cap V_1'$, and $x \in V_3 \cap V_3'$. By Corollary 13, $V(C) \subseteq AR_1(u) \cap AR_2(v) \cap AR_3(x)$ and $V(C') \subseteq AR_2(u) \cap AR_3(v) \cap AR_1(x)$ hold. Analogously to Case 1, we can show that $u \notin AR_1(v) \cup AR_1(x) \cup AR_2(x) \cup AR_3(x)$ and $v \notin AR_3(u) \cup AR_3(v) \cup AR_1(u) \cup AR_1(v)$ hold. Hence $u \in AR_3(v) \cap AR_3(x)$, $v \in AR_1(x) \cap AR_1(u)$, and $x \in AR_3(u) \cap AR_2(v)$ hold.

By Condition 9 and Lemma 10, either $AR_1(u) \cap AR_2(v) \cap AR_3(x) = \emptyset$ or $AR_2(u) \cap AR_3(v) \cap AR_1(x) = \emptyset$, which gives a contradiction. Consequently, Case 2 cannot occur, either. □

5 Constructing Region Graphs

In this section, we propose a procedure for inserting a new vertex in a given region graph preserving its conditions, that is, the output graph is a region graph including the new vertex. The procedure inserts the new vertex inside a prescribed region, and then constructs radial paths of such a vertex by following radial paths of vertices already placed in the graph.

Procedure \textsc{Insert}\textsubscript{H}(H, v, D, u)

\textbf{Input:} A region graph $H = (V, W, F)$, a new vertex $v$, a cycle $D$ of $H$ such that $V(D) \cup W(D) \subseteq AR_1(s_1) \cap AR_2(s_2) \cap AR_3(s_3)$ and there is no edges in the interior of $D$, and a vertex $u \in V(D) \cup W(D)$ such that if $u \in W(D)$, then $u$ is a common vertex of $P_1^-(u_1)$ and $P_1^-(s_1)$ for some $u_1 \in V$ and $i \in \{2, 3\}$.

\textbf{Output:} A new region graph $H' = (V' = V \cup \{v\}, W', F')$ such that $v$ is
placed in the interior of $D$.

**Case 1.** $u \in V(D)$. Assume that $V(D) \cup W(D) \subseteq R_i(u) \cap AR_j(u)$, where $i \neq j$ by the definitions of $R_i(u)$ and $AR_j(u)$. Let $k \in \{1, 2, 3\} - \{i, j\}$. Place $v$ inside $D$ so that it has the following six radial paths (see Fig. 8(a)).

Let $P_i^+(v)$ intersect with all radial paths which intersect with $P_i^-(u)$.

Let $P_i^+(v)$ intersect with $P_{j}^+(u)$ first, $P_k^-(u)$ second, and all radial paths which intersect with a subpath of $P_i^+(u)$.

Let $P_j^-(v)$ intersect with $P_i^-(u)$ first, $P_k^+(u)$ second, and all radial paths which intersect $P_j^-(u)$.

Let $P_j^+(v)$ intersect with all radial paths which intersect with $P_j^-(u)$.

Let $P_k^-(v)$ intersect with $P_j^+(u)$ first, and all radial paths which intersect with $P_k^-(u)$.

Let $P_k^+(v)$ intersect with $P_i^-(u)$ first, and all radial paths which intersect $P_k^+(u)$.

![Diagram](a)

![Diagram](b)

**Fig. 8.** Illustration of Procedure INSERT($H, v, D, u$). The grey region is $D$; (a) $u \in V(D)$; (b) $u \in W(D)$, where $u$ is a common vertex of $P_i^-(u_1)$ and $P_j^-(u_1)$.

**Case 2.** $u \in W(D)$. Traverse $P_i^-(u_1)$ from $u$ to $u_1$. Let $u' \in V$ denote a vertex such that a common vertex $x_1 \in W$ of $P_j(u')$ with $j \in \{2, 3\} - \{i\}$ and the subpath of $P_i^-(u_1)$ from $u$ to $u_1$ is nearest to $u$ in such common vertices. (If there is no such common vertices, then let $u_1 = x_1 = u'$). Let $x_2$ denote a common vertex of $P_j(u')$ and $P_i(s_1)$ if $P_j(u')$ and $P_i^-(s_1)$ intersect. Place $v$ inside $D$ so that it admits six radial paths of $v$ as follows (see Fig. 8(b)).

Let $P_i^-(v)$ intersect with all radial paths which intersect with the subpath of $P_i^-(u_1)$ from $u$ to $x_{1u_1}^-$.

Let $P_i^+(v)$ intersect with all radial paths which intersect with the subpath of $P_i^-(u_1)$ from $u$ to $u_1$ and $P_i^+(u_1)$.

Let $P_i^- (v)$ intersect with all radial paths which intersect with the subpath of $P_i^-(s_1)$ from $u$ to $x_{1}^s$.

Let $P_i^- (v)$ intersect with all radial paths which intersect with the subpath of
$P_i(s_1)$ from $u$ to $w_i^j$.

Let $P_{i'}(v)$ intersect with all radial paths which intersect with the subpath of $P_i^+(u_i)$ from $u$ to $x_1$ and the subpath of $P_i(u')$ from $x_1$ to $x_i^j$.

Let $P_{i'}(v)$ intersect with all radial paths which intersect with the subpath of $P_i(s_1)$ from $u$ to $x_2$ and a subpath of $P_j(u')$ from $x_2$ to $w_i^j$. \hfill $\square$

**Theorem 21** Procedure **INSERT**$(H, v, D, u)$ delivers a region graph $H'$ including $v$.

**Proof:** Assume that $u \in V(D)$. Obviously $H'$ satisfies Conditions 1-4. Note that, radial paths of $v$ are copies of radial paths of $u$. Hence, it follows directly that $H'$ satisfies the remaining conditions of region graphs since $H$ satisfies the same conditions.

Now, assume that $u \in W(D)$. Without loss of generality, we assume that $u$ is a common vertex of $P_i^-(u_i)$ and $P_i^-(s_1)$ for some $u_i \in V - \{s_1\}$. Hence $V(D) \subseteq ARD(u_i) \cap ARD(s_1)$ holds. Obviously $H'$ satisfies Conditions 1-6, 8, and 10. We claim that $H'$ also satisfies Conditions 7 and 9.

First, we show that $H'$ satisfies Condition 7. Since $H$ satisfies Condition 7, there is no radial path which intersects more than once with $P_i^-(v)$, $P_i^+(v)$, $P_i^-(v)$, or $P_i^+(v)$. Assume that a radial path $P$ intersects twice with $P_3^-(v)$. Then $P$ intersects with the subpath of $P_3^-(u_i)$ from $u$ to $x_1$ and the subpath of $P_3^-(u_i)$ from $x_1$ to $x_i^j$. Hence $P$ is a green radial path, where $P$ is $P_i^-(u_i)$ or $P_i^-(u_i)$ for some $u_i \in V$. If $u_i = u_i$, then by Condition 4 in $H$, $P$ intersects with the green path of $u_i$, which contradicts Condition 5 in $H$. Assume that $u_i \neq u_i$.

We distinguish the following two cases.

(i) $P_i^-(u_i)$ intersects with $P_i^-(u_i)$. Then there is a region bounded by $P_i^-(u_i)$, $P_i^-(u_i)$, and $P_i^+(u_i)$ (see Fig. 9(a)). Therefore, $P$ passes such a region and intersects twice with $P_i^-(u_i)$ or intersects with $P_i^+(u_i)$, which contradicts Conditions 7 and 5 in $H$, respectively.

(ii) $P_i^+(u_i)$ intersects with $P_i^-(u_i)$. Then $u_i \in R_2(u_i)$ and either $u_i \in R_2(u_i)$ or $u_i \in R_3(u_i)$ holds. If $u_i \in R_2(u_i)$, then by Lemma 10, $P_i^-(u_i)$ intersects with $P_i^+(u_i)$ and hence there is a region bounded by $P_i^-(u_i)$, $P_i^+(u_i)$, and $P_i^+(u_i)$ (see Fig. 9(b)). If $u_i \in R_2(u_i)$, then $P_i^+(u_i)$ intersects with $P_i^+(u_i)$ and hence there is a region bounded by $P_i^-(u_i)$, $P_i^-(u_i)$, and $P_i^+(u_i)$ (see Fig. 9(c)). In both cases, we can show that $H'$ satisfies Condition 7 in the same way with (i).

Finally, we show that $H'$ satisfies Condition 9. We prove that for each $u \in V$, the number of common vertices between \{ $P_i^-(v)$, $P_i^-(v)$, $P_i^-(v)$ \} and \{ $P_i^-(u)$, $P_i^-(u)$, $P_i^-(u)$ \} is exactly one. By the construction of radial paths of $v$, we observe that Lemma 6(iv)-(v) holds in $H'$. Hence if the number of common
Fig. 9. Illustration of Theorem 21: (a) $P_3^-(u')$ intersects with $P_1^-(u_1)$; (b) $P_3^+(u')$ intersects with $P_1^-(u_1)$ and $u' \in R_1(u_1)$; (c) $P_3^+(u')$ intersects with $P_1^-(u_1)$ and $u' \in R_3(u_1)$.

vertices between $\{P_1^-(v), P_2^-(v), P_3^-(v)\}$ and $\{P_1^-(u), P_2^-(u), P_3^-(u)\}$ is more than one, then two paths of the same color intersect, which contradicts Condition 5 in $H'$. Therefore $H'$ satisfies Condition 9. □

6 Consistent Region Graphs

Recall that a region graph $H = (V, W, F)$ is introduced as a new data structure which can play the same role of directional $S$-embeddings for partitioning a vertex set $V$ into subsets $V_1, V_2$ and $V_3$ with prescribed sizes. In this section, we first show how to construct a region graph $H = (V, W, F)$ for a given triconnected graph $G = (V, E)$ with $S = \{s_1, s_2, s_3\} \subseteq V$ in order to ensure that each subset $V_i$ induces a connected subgraph from $G$. Given a triconnected graph $G = (V, E)$ and a subset $S \subseteq V$ with $|S| = 3$, a region graph $H = (V, W, F)$ is called consistent if, for each $v \in V - S$ and $i = 1, 2, 3$, it holds $N(v; G) \cap R_i(v) \neq \emptyset$ (see Fig. 10(a)-(b)).

Lemma 22 For a triconnected graph $G = (V, E)$ and a set $S = \{s_1, s_2, s_3\}$ of three vertices of $V$, let $H = (V, W, F)$ be a consistent region graph. Then, for any ordered partition $(V_1, V_2, V_3)$ of $V$ defined by a cell $C$ of $H$, if $V_i \neq \emptyset$ holds, then $V_i$ induces a connected subgraph of $G$ containing $s_i$, $i = 1, 2, 3$.

Proof: Without loss of generality we prove the lemma for $i = 1$. Assume that $V_1 \neq \emptyset$. By Lemma 18, $s_1 \in V_1$ holds. For each vertex $v \in V_1 - \{s_1\}$, we show that there is a path $P$ from $v$ to $s_1$ such that $V(P) \subseteq V_1$. By the definition of consistent region graphs, there is a vertex $u \in N(v; G) \cap R_1(v)$ and hence $u \in V_1$ by Lemma 18. Since $u \in R_i(v)$, it holds $R_i(u) \subseteq R_i(v)$ by Lemma 11(1). Recursively, we can find a path $P'$ from $u$ to $s_1$ such that $V(P') \subseteq V_1$. Hence, $(v, u)$ and $P'$ form a desired path $P$ that reaches $s_1$. □
Fig. 10. (a) A triconnected graph $G = (V, E)$ with a set $S = \{s_1, s_2, s_3\} \subseteq V$; (b) A consistent region graph $H = (V, W, F)$ for $G$ and $S$; (c) A consistent region graph without all $c$-edges, where each inner face corresponds to a cell.

**Theorem 23** Given a consistent region graph for a triconnected graph $G = (V, E)$ and a set $S$ of three vertices of $V$, a tripartition for $G$ and $S$ can be computed in $O(|V|)$ time.

**Proof:** By Theorem 19, for any natural numbers $n_1$, $n_2$, and $n_3$ with $n_1 + n_2 + n_3 = |V|$, we can find an ordered partition $(V_1, V_2, V_3)$ of $V$ such that $|V_i| = n_i$, $i = 1, 2, 3$, in $O(|V|)$ time. Moreover, by Lemma 22, $V_i$ induces a connected subgraph of $G$ containing $s_i$ for all $i = 1, 2, 3$. This completes the proof. □

In the rest of this section, we propose an algorithm for constructing a consistent region graph for a given triconnected graph $G$. The algorithm contraction and/or removal of edges of the graph preserving triconnectivity until we get a triangle (a cycle with length 3), and then constructs a consistent region graph of this triangle. By backtracking the sequence of contractions and removals of edges, we construct a desired consistent region graph by recursively using the procedure described in the previous section.

**Lemma 24** (Tutte [22]) Let $G = (V, E)$ be a triconnected graph. For any edge $e \in E$, either $G/e$ is triconnected or $G - e$ is a subdivision of a triconnected graph.

Note that, it can be checked in linear time whether a given graph $G$ is triconnected or not [10]. On the other hand, for a triconnected graph $G$ and an edge $e = (v_1, v_2)$ such that $G - e$ is a subdivision of a triconnected graph $G'$, such a triconnected graph $G'$ can be constructed as follows. For each vertex $v_i$, $i = 1, 2$, if the degree of $v_i$ becomes 2, then for the two neighbours $x_i$ and $y_i$ of $v_i$, remove edges $(v_i, x_i)$, $(v_i, y_i)$, and the vertex $v_i$ after adding an edge $(x_i, y_i)$ (if $(x_i, y_i)$ is not an edge of $G$).
Lemma 25  Let $G = (V, E)$ be a triconnected graph which contains a triangle with vertex set \{s_1, s_2, s_3\} such that $s_1$ has only one neighbour $v \in V - \{s_2, s_3\}$ in $G$. Then $G/(s_1, v)$ is triconnected.

Proof: Assume for the contrary that $G/e$ is not triconnected. Then for some $u_1, u_2 \in V$, there does not exist a set of three vertex-disjoint paths from $u_1$ to $u_2$ in $G/e$. Since $G$ is triconnected, there exist three vertex-disjoint paths $P_1$, $P_2$, and $P_3$ from $u_1$ to $u_2$ in $G$ and without loss of generality we can assume that $v \in V(P_1)$ and $s_1 \in V(P_2)$ since $G/e$ is not triconnected. Therefore, $s_2$, $s_3$ $\in V(P_2)$ since $s_1$ has only one neighbour $v \in V - \{s_2, s_3\}$. Then there exist three vertex-disjoint paths $P_1$, $P_2$, and $P_3$ such that $P_2'$ is obtained from $P_2$ by exchanging edges $(s_2, s_1)$ and $(s_1, s_3)$ for the edge $(s_2, s_3)$, a contradiction. Hence $G/e$ is triconnected. \hspace{1cm} \Box

Now we describe our algorithm CRG for finding a consistent region graph. Note that, given a triconnected graph $G = (V, E)$, a sparse spanning triconnected subgraph $G'$ of $G$ with $V(G') = V(G)$ and $O(n)$ edges can be computed in linear time [15]. Therefore, we can assume without loss of generality that the triconnected graph given in algorithm CRG has $O(n)$ edges.

Algorithm CRG

Input: A triconnected graph $G = (V, E)$ and a set $S = \{s_1, s_2, s_3\}$ of three vertices of $V$.

Output: A consistent region graph $H = (V, W, F)$ for $G$ and $S$.

Phase 1:

Add an edge $(s_i, s_j)$ to $G$ if $(s_i, s_j) \notin E, i, j \in \{1, 2, 3\}$. While there exists a neighbour $v \in V - \{s_2, s_3\}$ of $s_1$ do

   Apply one of following (i) and (ii):

   (i) Let $G := G/(s_1, v)$ if $G/(s_1, v)$ is triconnected.

   (ii) Let $G := G - (s_1, v)$, and if the degree of $v$ in $G$ is 2, then for the two neighbours $u$ and $w$ of $v$, remove edges $(u, v), (v, w)$ and vertex $v$, and add an edge $(u, w)$ (if there does not exist).

Phase 2:

Let $G_0 = G, G_1, \ldots, G_{\ell}$ be the resulting sequence of graphs obtained in Phase 1, where $G_0$ is obtained after $\ell$th iteration of the while-loop.

Find a consistent region graph $H_\ell$ of $G_\ell$.

For $q = \ell, \ell - 1, \ldots, 1$, apply one of the following (i), (ii), and (iii).

   (i) If $G_q$ is obtained from $G_{q-1}$ by removing an edge $e = (s_1, v)$, then $H_{q-1} := H_q$, which is a consistent region graph for $G_{q-1}$ and $S$.

   (ii) If $G_q$ is obtained from $G_{q-1}$ by removing $(s_1, v), (u, v), (v, w)$, and $v$, and adding an edge $(u, w)$, then apply one of the following (1) and (2).

1. If $w \in R_3(u)$ (the case $u \in R_3(w)$ can be treated analogously),
then let $D$ be a cycle containing $u$ and edges of $P_{3}^- (u)$ and $P_{2}^+ (u)$ but containing no edges in its interior (see Fig. 11(a)). Construct a region graph $H_{q-1}$ of $G_{q-1}$ by Procedure INSERT($H_{q}, v, D, u$).

2. If $u \in R_{2}(w)$ and $w \in R_{1}(u)$ (the case $u \in R_{1}(w)$ and $w \in R_{2}(u)$ can be treated analogously), then let $D$ be a cycle containing $w$ and edges of $P_{3}^- (w)$ and $P_{2}^+ (w)$ but containing no edges in its interior (see Fig. 11(b)). Construct a region graph $H_{q-1}$ of $G_{q-1}$ by Procedure INSERT($H_{q}, v, D, w$).

(iii) If $G_q$ is obtained from $G_{q-1}$ by contracting $e = (s_1, v)$, then apply one of the following (1) and (2).

1. If $(N(v; G_{q-1}) - \{ s_1 \}) \cap R_i(s_1) \neq \emptyset$ for all $i \in \{2, 3\}$, then let $D$ be a cycle containing $s_1$ and edges of $P_1^+ (s_1)$ and $P_2^- (s_1)$ but containing no edges in its interior (see Fig. 12(a)). Construct a region graph $H_{q-1}$ of $G_{q-1}$ by Procedure INSERT($H_{q}, v, D, s_1$).

2. If $(N(v; G_{q-1}) - \{ s_1 \}) \cap R_i(s_1) = \emptyset$ for some $i \in \{2, 3\}$, then traverse $P_2^-(s_1)$ with $j \in \{2, 3\} - \{ i \}$ from $s_1$ to $x_{s_1}^i$ and look for a common vertex of $P_j^-(s_1)$ and $P_1^+(u)$ for each $u \in N(v; G_{q-1}) - \{ s_1 \}$. Let $w$ be the nearest vertex to $s_1$ among such common vertices, and let $D$ be a cycle containing $w$, $P_2^-(s_1)$, and $P_1^- (u)$, but containing no edges in its interior (see Fig. 12(b)). Construct a region graph $H_{q-1}$ of $G_{q-1}$ by Procedure INSERT($H_{q}, v, D, w$) (see Fig. 12(c)).

Output a consistent region graph $H = H_0$. □

![Fig. 11. Illustration of Algorithm CRG (Phase 2(ii)); (a) $w \in R_3(u)$; (b) $u \in R_2(w)$ and $w \in R_1(u)$.](image)

Note that, a region graph of a triangle with vertex set $\{ s_1, s_2, s_3 \}$ can be computed by first drawing its boundary $B$ as a circle, inserting vertices $s_1$, $s_2$, and $s_3$ inside $B$, and then drawing radial paths of these vertices such that $P_1^+ (s_j)$ intersects with $P_j^- (s_i)$ for all $i, j \in \{1, 2, 3\}$ (see Fig. 3(b)).

**Theorem 26** A region graph $H$ output by Algorithm CRG is consistent for $G$ and $S$.  

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Fig. 12. Illustration of Algorithm CRG (Phase 2(iii)). Black vertices denote the vertices in $N(v; G_{q-1})$: (a) $(N(v; G_{q-1}) - \{s_1\}) \cap R_i(s_1) \neq \emptyset$, $i = 2, 3$; (b) $(N(v; G_{q-1}) - \{s_1\}) \cap R_3(s_1) = \emptyset$; (c) A region graph $H_{q-1}$ output by Procedure $\text{INSERT}(H_q, v, D, w)$ applied to the instance in (b).

**Proof:** Obviously, the correctness of Phase 1 follows from Lemmas 24 and 25. Now, we consider Phase 2. It is sufficient to prove the correctness of an arbitrary iteration of Phase 2. For some $1 \leq q \leq \ell$, let graph $G_q$ be obtained from $G_{q-1}$ by one iteration of Phase 1, and let $H_q$ be a consistent region graph for $G_q$ and $S$. We show that the graph $H_{q-1}$ obtained from $H_q$ is a consistent region graph for $G_{q-1}$ and $S$. If we obtain $H_{q-1}$ by applying (i) to $H_q$, then obviously $H_{q-1}$ is a consistent region graph for $G_{q-1}$ and $S$.

Assume that $H_{q-1}$ is obtained by applying (ii) to $H_q$. Assume that $w \in R_3(u)$ and $u \in R_2(w)$ in $H_q$. By Theorem 21, $H_{q-1}$ is a region graph. Since $v \in AR_1(s_1) \cap AR_2(u) \cap AR_3(w)$ in $H_{q-1}$, it follows from Lemma 10 that $s_1 \in R_1(v)$, $u \in R_2(v)$ and $w \in R_3(v)$ in $H_{q-1}$. This implies that $N(v; G_{q-1}) \cap R_i(v) \neq \emptyset$ holds in $H_{q-1}$ for all $i = 1, 2, 3$. Hence $H_{q-1}$ is a consistent region graph for $G_{q-1}$ and $S$. Analogously, we can prove that $H_{q-1}$ is a consistent region graph for $G_{q-1}$ and $S$ when $w \in R_3(u)$ and $u \in R_1(w)$ or $w \in R_1(u)$ and $u \in R_3(w)$ in $H_q$.

Now, assume that $H_{q-1}$ is obtained by applying (iii)(1) to $H_q$. That is, $(N(v; G_{q-1}) - \{s_1\}) \cap R_i(s_1) \neq \emptyset$, $i = 2, 3$, holds. Since $v \in AR_1(s_1)$, it follows from Lemma 10 that $s_1 \in R_1(v)$ holds. Note that, for each $x \in (N(v; G_{q-1}) - \{s_1\}) \cap R_2(s_1)$, $v \in AR_2(x) \cap R_1(x)$ since $D$ contains no edges in its interior, and hence Lemma 10 implies that $x \in R_2(v)$ holds. Analogously, we can show that, for each $y \in (N(v; G_{q-1}) - \{s_1\}) \cap R_3(s_1)$, it holds $y \in R_3(v)$. Therefore, for each $i = 1, 2, 3$, it holds $N(v; G_{q-1}) \cap R_i(v) \neq \emptyset$ in $H_{q-1}$. Hence $H_{q-1}$ is a consistent region graph for $G_{q-1}$ and $S$.

Finally, assume that $H_{q-1}$ is obtained by applying (iii)(2) to $H_q$. Without loss of generality we assume that $(N(v; G_{q-1}) - \{s_1\}) \cap R_3(s_1) = \emptyset$. Let $w$ be the nearest common vertex of $P^-_2(s_1)$ and $P^-_3(u)$ to $s_1$. Since $v \in AR_1(s_1) \cap AR_3(u)$, it follows from Lemma 10 that $s_1 \in R_1(v)$ and $u \in R_3(v)$ hold. Note
that, for each \(x \in (N(v; G_{q-1}) - \{s_1, u\})\), \(v \in AR_2(x) \cap R_1(x)\) since \(D\) contains no edges in its interior, and hence Lemma 10 implies that \(x \in R_2(v)\) holds. Therefore, for each \(i = 1, 2, 3\), it holds \(N(v; G_{q-1}) \cap R_i(v) \neq \emptyset\) in \(H_{q-1}\). Hence \(H_{q-1}\) is a consistent region graph for \(G_{q-1}\) and \(S\). \(\Box\)

**Theorem 27** Let \(G = (V, E)\) be a triconnected graph and \(S \subseteq V\) be a set of three vertices of \(V\). Then Algorithm CRG constructs a consistent region graph for \(G\) and \(S\) in \(O(|V|^2)\) time.

**Proof:** Executing each iteration in Phase 1 of Algorithm CRG takes \(O(|V|)\) time since triconnectivity can be tested in linear time. Therefore, Phase 1 runs in \(O(|V|^2)\) time since \(G\) has \(O(|V|)\) edges. Now, consider the procedure of inserting a new vertex in the current consistent region graph to construct a new consistent region graph containing such a vertex. Radial paths of such a new vertex intersect at most \(O(|V|)\) radial paths of the existent vertices. Hence a new consistent region graph is constructed in \(O(|V|)\) time. This implies that Phase 2 also can be executed in \(O(|V|^2)\) time. Therefore, we can construct a consistent region graph for \(G\) and \(S\) in \(O(|V|^2)\) time. \(\Box\)

7 Conclusion

In this paper, we have introduced consistent region graphs as a new data structure that represents a given triconnected graph \(G = (V, E)\) and a specified set \(S\) of three vertices of \(V\). We have designed an \(O(|V|^2)\) time algorithm for constructing a consistent region graph. Given a consistent region graph, we can find a tripartition in linear time. Moreover, we easily see that 3-independent trees for \(G\) and \(S\) can be derived from a consistent region graph. It would be interesting to construct efficient data structures which enable us to find \(k\)-partitions of \(k\)-connected graphs for \(k \geq 4\).

References


