Extending Steinitz’ Theorem to Non-convex Polyhedra

Seok-Hee Hong
School of Information Technologies
University of Sydney
shhong@it.usyd.edu.au

Hiroshi Nagamochi*
Graduate School of Informatics
Kyoto University
nag@amp.i.kyoto-u.ac.jp

Abstract: The famous Steinitz’ theorem of 1930’s gives a complete characterization of the topological structure of the vertices, edges and faces of convex polyhedra as triconnected planar graphs. In this paper we generalize Steinitz’ theorem. We introduce upward polyhedra, which are defined as polyhedra such that each face is star-shaped, all faces except the bottom face are visible from a viewpoint, and any two faces sharing two vertices are non-coplanar. We give a graph-theoretic characterization of upward polyhedra; roughly speaking, they correspond to bi-connected planar graphs with some extra conditions. The proof of the characterization leads to an algorithm that constructs an upward polyhedron with $n$ vertices in $O(n^{5/3})$ time. Moreover, one can test whether a given plane graph is an upward polyhedral graph in linear time. This is the first graph-theoretic characterization of polyhedra that are not necessarily convex.

1 Introduction

Polyhedra appear in early history, architecture, art, ornament, nature, cartography and philosophy. For detailed examples, see Cromwell [15] and Coxeter [12]. There is a rich literature on polytopes in mathematics, in particular convex polytopes, see Grünbaum [23] and Kalai [38, 39].

The most well-known Steinitz’ theorem [56, 57] (see Theorem 1), back to 1920s, characterizes a simple convex 3-polytope as a simple triconnected planar graph. Many researchers investigated polytope (or polytopal) graphs, i.e. the graphs of polytopes, mainly for convex polytopes, see Grünbaum [23, 24, 25] and Kalai [40].

In fact, there is a strong relationship between graphs and polytopes. Perles [50] conjectured, and Blind and Mani [6] proved that a simple polytope is uniquely determined by its 1-skeleton graph. For a simpler proof, see Kalai [37], “A simple way to tell a simple polytope from its graph.”

Note that many examples of “non-convex polyhedra” are available, see [15, 22, 26, 28, 27, 41, 46, 51, 55]. Examples of non-convex polyhedra include star-shaped polyhedra [27], acyclic polyhedra [26] and some of uniform polyhedra [12, 15].

However, as pointed out by Grünbaum [29] in his recent survey on graphs and polyhedra in 2007, non-convex polyhedra have not been well studied, in particular, the polyhedral graphs of non-convex polyhedra. He addressed cross-fertilization of the geometric and combinatorial aspects of more general polyhedra, as new research direction, and wrote “the most central obstacle to any coherent theory of polyhedra more general than the convex one is the difficulty of defining precisely what objects should be awarded that designation.” In this paper, as the first step for discovering such a new class of polyhedra which admits a graph-theoretic characterization, we introduce a new wider class of polyhedra, called upward polyhedra (formally defined in Section 2.1). We prove that upward polyhedra admits a complete graph-theoretic characterization by presenting a constructive proof (see Theorem 5).

1.1 Related Work

Convex drawings of graphs in both two and three dimensions are well established problems. Convex drawings in two dimensions have been extensively studied, see [7, 8, 9, 10, 47, 53, 58, 61, 62]. Tutte [61, 62] gave a necessary and sufficient condition for a plane graph to admit a convex drawing.*

He showed that every triconnected plane graph with a given boundary drawn as a convex polygon admits a convex drawing using the polygonal boundary, and proposed a “barycenter mapping” method which computes a convex drawing of a triconnected plane graph. Thomassen [58] gave a necessary and sufficient condition for a biconnected plane graph to admit a convex drawing, and Chiba et al. [8] gave a linear time algorithm for finding a convex drawing for a biconnected plane graph with a specified convex boundary. Convex grid drawings are studied by Bonichon et al. [7], Chrobak et al. [9], Chrobak and Kant [10], and Miura et al. [47].

Convex drawings of graphs in three dimensions are well known as Steinitz’ theorem (see Theorem 1). Many proofs of Steinitz’ theorem are available [3, 23, 56, 57, 60]. One can devise a naive cubic time algorithm from the constructive proofs, however, it may produce a polytope with exponential volume. Faster $O(n^{1.5})$ time algorithms are available by Hopcroft and Kahn [33], Eades and Garvan [21], and Chrobak, Goodrich and Tamassia [9], where $n$ is the number of vertices. They are all based on the notion of convex equilibrium stress graphs, dating back to Maxwell [45] (see also Connelly [11], Crapo and Whiteley [13, 14], and Whiteley [64, 65]).

The algorithm by Eades and Garvan is based on the Tutte’s barycenter method [61, 62], however, the resulting polytope may have exponential volume. Chrobak, Goodrich and Tamassia [9] achieved polynomial volume, based on the formulation defined by Hopcroft and Kahn [33]. Further, their algorithm can be implemented to run in $O(n^{1.19})$ at best, however in practice in runs in $O(n^{1.5})$. For triangulated planar graphs, there is a linear time algorithm by Das and Goodrich [16].

Some extensions of Steinitz’ theorem are available, see Grünbaum [25], and Klee and Kleinschmidt [42]. Mani [44] proved that the full automorphism group of a triconnected planar graph can be realized as symmetries of a convex polytope (for another proof, see Schramm [54]). See Barnette and Grünbaum [4] for preassigning the shape of a face.

The stronger version of Steinitz’ theorem is known as Koebe-Andreev-Thurston theorem [43, 2, 59] (also known as the circle packing theorem or the canonical polytope problem), see Ziegler [66], and Demaine and Erikson [17]. Although it is possible to construct such a polytope using numerical method such as the algorithm by Mohar [48, 49] in polynomial time, the pure combinatorial algorithm is still open. The algorithm by Hart [31] works in most cases, however fails in some cases [29]. The quasi convex programming method by Bern and Eppstein [5] runs in linear time to construct the canonical sphere drawing.

Recently, characterizations (together with recognition algorithms) are given for special subclasses of convex polyhedra including equiprojective polyhedra [32] and zonohedra [1]. Note that these are special types of regular convex polyhedra.

The celebrated theorem by Blind and Mani [6] proved that the vertex-edge graph $G(P)$ of a simple $d$-dimensional polytope $P$ determines the entire combinatorial structure of $P$. However, their proof is not constructive. Kalai [37] gave a very short, elegant and constructive proof, however, his algorithm has exponential running time. For various methods for faster implementation of Kalai’s proof, see [35, 36]. See also [52] for polytope reconstruction.

1.2 Our Results and Contribution

We now briefly describe our main results and contribution.

(1) We introduce a new class of polyhedra, called upward polyhedra (see Section 2.1 for a formal definition), which admits a graph-theoretic characterization. Our upward polyhedra have an application in Graph Drawing [20], Computational Geometry and Computer Graphics (such as morphing, animation, and molding), and Machine Interpretation of pictures [55]. It allows a view point which can see all the vertices and edges of the graph of the polyhedron. Thus, we can guarantee the full visibility of the polyhedron without any occlusion problem.

(2) We present a complete graph-theoretic characterization of upward polyhedral graphs. Note that there are biconnected plane graphs with degree at least 3 that have no realization as the vertex-edge graphs of nonsingular polyhedra (see Lemma 3). In fact, our main theorem (Theorem 5) implies Steinitz’ theorem as a special case. To our best knowledge, our work is the first graph-theoretic characterization on polyhedra not necessarily convex.
(3) Our constructive proof gives an algorithm for constructing an upward polyhedron in $O(n + T(n))$ time. $T(n)$ denotes the running time of an algorithm for implementing Steinitz’ theorem. The current best runs in $O(n^{1.5})$ time [9, 21].

(4) We can test whether a given plane graph is an upward polyhedral graph in linear time.

2 Preliminaries

For notational simplicity, the facial cycle (resp., the simple-connected region) of a face $f$ may be referred to as the cycle $f$ (resp., the region $f$). Similarly, unless confusion arises, we use the name of a vertex (resp., edge and face) of a graph/polyhedron as the position of a corresponding vertex (resp., the line segment of a corresponding edge, and the region of a corresponding face) in $\mathbb{R}^3$.

2.1 Upward Polyhedra

For a point $p$ in the three dimensional Euclidean space and a positive $\varepsilon > 0$, let Ball($p, \varepsilon$) denote the ball with center $p$ and radius $\varepsilon$. For a polyhedron $P$, a point $p$ is called an interior point (resp., exterior point) if Ball($p, \varepsilon$) $\subseteq P$ (resp., Ball($p, \varepsilon$) $\cap P = \emptyset$) for some $\varepsilon > 0$, and is called a boundary point if Ball($p, \varepsilon$) $\nsubseteq P$ for any $\varepsilon > 0$. Let Int($P$) (resp., Ext($P$)) denote the set of all interior (resp., exterior) points of $P$ and let $\partial(P)$ denote the set of all boundary points of $P$. In this paper, a point is called a nonsingular point of a polyhedron $P$ if there is a positive $\varepsilon_0$ such that each of Int($P$) $\cap$ Ball($p, \varepsilon$) and Ext($P$) $\cap$ Ball($p, \varepsilon$) is nonempty and connected for all $\varepsilon \in (0, \varepsilon_0)$.

A point is called a singular point of a polyhedron $P$ if there is no such $\varepsilon_0$. We consider nonsingular polyhedra without holes in the three dimensional Euclidean space $\mathbb{R}^3$. We define “polyhedra” in $\mathbb{R}^3$ mostly according to Grünbaum [26]. A polyhedron is a collection of planar, compact, simply-connected polygonal regions; the boundary of such a region is called a simple polygon and the region itself is referred as a face of the polyhedron. A simple polygon consists of a finite number of line segments of positive length (the edges of the polygon); the endpoints of the edges are the vertices of the polygon. Each vertex of a polygon belongs to precisely two edges (said to be mutually adjacent), and the edges form a simple circuit (Jordan polygon). It should be noted that we made no assumption of convexity, and that edges (adjacent or nonadjacent) may be collinear. We assume that each edge is shared by precisely two faces and that all faces containing a given vertex form a single circuit of at least three faces. A polyhedron is called acyclic [26] if the relative interiors of its elements (faces, edges and vertices) are disjoint.

In this paper, we call an acyclic polyhedron a spherical polyhedron if it has no singular points and no holes, and two faces sharing an edge are not coplanar (i.e., they are in different planes). Spherical polyhedra satisfy Euler’s formula which gives a combinatorial relationship between the numbers of vertices $n$, edges $m$ and faces $f$: $n - m - f = 2$. A given set of vertices, edges and faces which satisfies this formula forms a topological sphere, a combinatorial embedding of a planar graph. We call such a planar graph of a spherical polyhedron $P$ the vertex-edge graph of $P$, denoted by $G(P)$. The plane that contains a face of a spherical polyhedron is called supporting if it cuts the three-dimensional space into two half-spaces, one of which contains the entire polyhedron. A face contained in a supporting plane is called supporting.

The surface of a spherical polyhedron separates the three dimensional space into the surface, the bounded interior and unbounded exterior of the polyhedron. An edge of a spherical polyhedron is called convex if the interior angle made by the two faces that share the edge is less than $\pi$, and is called concave otherwise (i.e., the interior angle is greater than $\pi$). A convex polyhedron is a spherical polyhedron in which all faces are convex polygons and all edges are convex. A convex polyhedron can also be described as a spherical polyhedron in which all faces are supporting.

A natural question is: which planar graphs (with the regions formed in the plane taken as the faces) can be realized as the faces, edges and vertices of a spherical polyhedron? For the case of convex polyhedra, this question was completely settled by Steinitz [56, 57] in 1920s, who gave the following combinatorial characterization of all those graphs.
Theorem 1 (Steinitz) A graph $G$ is the vertex-edge graph of a convex polyhedra $P$ if and only if it is a triconnected planar graph.

However, no graph-theoretic characterization is known for a planar graph $G$ with specified corners such that there is a spherical polyhedron $P$ with $P(G) = G$ in which the specified corners are realized as concave corners, even if $G$ is triconnected. Note that characterizing general spherical polyhedra is as hard as characterizing spherical polyhedra with the concave corner constraint (i.e., specified corners need to be realized as concave corners), see Appendix A.

In this paper, we introduce “upward” polyhedra, a wider class of spherical polyhedra. A spherical polyhedron $P$ is called upward-visible if there is a supporting face $f$ such that the image of $G(P)$ by a parallel projection to the supporting plane $H_f$ of $f$ is a straight-line plane drawing of $G(P)$ with the outer face $f$. We call such a face $f$ the bottom face of $P$. All faces except the bottom face are visible from a view point over $P$.

A spherical polyhedron $P$ is called crown-shaped if it has a face $f_b$ (called the bottom face) and a view point $p$ in the exterior of $P$ such that the exterior surfaces (or the interior surfaces) of all faces except for $f_b$ are visible, where we assume that the interior surfaces can be seen through the face $f_b$ (see Fig. 1).

Let $P$ be a spherical polyhedron. We say that two nonadjacent faces which share two or more vertices have a bend in $P$ if they are not coplanar in $P$, and that $P$ is strictly bended if all pairs of such faces have bends. In a strictly bended polyhedron, two faces sharing two vertices need to be non-coplanar even if they do not share any edge (Fig. 21(c) shows a spherical polyhedron which is not strictly bended).

A spherical polyhedron $P$ is called upward if it is a crown-shaped, strictly bended polyhedron with star-shaped faces. In what follows, we only consider the case where an upward polyhedron is upward-visible.

2.2 Planar/Plane Graphs, Graph Connectivity and Decomposition

Throughout the paper, a graph stands for a simple undirected graph. Let $G = (V, E)$ be a graph. The minimum degree of a vertex in $G$ is denoted by $\delta(G)$.

A vertex in a connected graph is called a cut vertex if its removal from $G$ results in a disconnected graph. A connected graph is called biconnected if it is simple and has no cut vertex. Similarly, a pair of vertices in a connected graph is called a cut pair (or separation pair) if its removal from $G$ results in a disconnected graph. A connected graph is called triconnected if it is simple and has no cut pair. In general, a graph $G$ with more than $k$ vertices is called $k$-connected if the graph remains connected after removing any $k - 1$ vertices together with the incident edges. The vertex-connectivity $\kappa(G)$ of a graph is the maximum integer $k$ such that $G$ is $k$-connected. A cycle $C$ is $k$-connected in $G$ if $G$ has a vertex $v$ not in $C$ and $k$ vertices $u_1, u_2, \ldots, u_k$ in $C$ such that $v$ has $k$ internally vertex-disjoint paths $P_i$, $i = 1, 2, \ldots, k$, each joins $v$ and $u_i$.

A graph $G = (V, E)$ is called planar if its vertices and edges are drawn as points and curves in the plane so that no two curves intersect except for their endpoints, where no two vertices are drawn at the same point. In such a drawing, the plane is divided into several connected regions, each of which is called a face. A face is characterized by the cycle of $G$ that surrounds the region. Such a cycle is called a facial cycle. A set $F$ of facial cycles in a drawing is called a combinatorial embedding of a planar graph $G$.

A plane graph $G = (V, E, F)$ is a planar graph $G = (V, E)$ with a fixed plane embedding $F$ of $G$, where we always denote the fixed outer facial cycle in $F$ by $f_o$. A vertex (resp., an edge) in $f_o$ is called an outer vertex (resp., an outer edge), while a vertex (resp., an edge) not in $f_o$ is called an inner vertex (resp., an inner edge). The following lemma describes that every facial cycle of $G(P)$ of a spherical polyhedron $P$ is triconnected (see Appendix D.3 for a proof).

Lemma 2 Let $G = (V, E, F)$ be a biconnected planar graph with a fixed embedding. If there is a spherical polyhedron $P$ with $G(P) = G$, then each cycle $f \in F$ is triconnected.
A straight-line drawing of a graph \( G = (V, E) \) in the plane is an embedding of \( G \) in the two-dimensional space \( \mathbb{R}^2 \), such that each vertex \( v \in V \) is drawn as a point in \( \mathbb{R}^2 \), and each edge \( (u, v) \in E \) is drawn as a straight-line segment \([u, v] \). A straight-line drawing is called a straight-line plane drawing if no straight-line segment intersects with other straight-line segment at its internal point. A polygon is called star-shaped if it contains an internal point \( p^* \) from which any point \( p \) on the boundary of the polygon is visible (i.e., the line-segment \([p^*, p]\) contains no other point on the boundary of the polygon).

Let \( G = (V, E, F) \) be a biconnected simple plane graph with the outer face \( f^o \). We say that two faces \( f \) and \( f' \) are adjacent if they share an edge. For a pair \( \{u, v\} \) of vertices, two faces \( f \) and \( f' \) are joined at \( \{u, v\} \) if they share the two vertices \( u \) and \( v \). We say that two faces \( f \) and \( f' \) are linearly-joined if they share an edge or at least three vertices (note that they are not linearly-joined if they share exactly two vertices but no edge). The following lemma describes a forbidden structure for biconnected plane graphs \( G \) to be realized as spherical polyhedra (see Appendix D.1 for a proof).

**Lemma 3** Let \( G = (V, E, F) \) be a biconnected planar graph with a fixed embedding. If \( G \) has a separation pair \( \{u, v\} \) and three edges \( (u, w_i), i = 1, 2, 3 \), incident to \( u \) such that each edge \((u, w_i)\) is shared by a pair of linearly-joined faces at \( \{u, v\} \), then there is no spherical polyhedron \( P \) with \( G(P) = G \).

The splitness \( \xi(u, v) \) of a pair \( \{u, v\} \) of vertices is defined by the number of pairs of linearly-joined inner faces joined at \( \{u, v\} \). We define the splitness \( \xi(G) \) of a plane graph \( G \) as the maximum of \( \max\{\xi(u, v) + 1 \mid u \text{ and } v \text{ are outer vertices}\} \) and \( \max\{\xi(u, v) \mid \text{one of } u \text{ and } v \text{ is an inner vertex}\} \). Fig. 2 illustrates the structure of plane graphs \( G \) with \( \xi(G) \geq 2 \).

We now briefly review the decomposition of a biconnected graph \( G \) into triconnected components [34], which will be the basis of a constructive proof for our main theorem. Tutte [63] firstly defined that the decomposition has a tree structure, now known as the 3-block tree or the SPQR tree by di Battista and Tamassia [18, 19]. For notational simplicity, we use the terminology from the SPQR tree. See Appendix B for details for the decomposition and the SPQR tree.

Each node \( \nu \) in the SPQR tree is associated with a graph \( \sigma(\nu) = (V_\nu, E_\nu) \) \((V_\nu \subseteq V)\), called the skeleton of \( \nu \), which corresponds to a triconnected component of \( G \). In fact, we use a modified SPQR tree without Q-nodes. Thus, there are only three types of nodes in the SPQR tree: S-node (the skeleton is a simple cycle), P-node (the skeleton consists of two vertices with at least 3 edges), and R-node (the skeleton is a triconnected simple graph). The edges of the SPQR tree are defined by the virtual edges which were introduced in the decomposition process. If two triconnected components have a virtual edge in common, then the nodes that represent the two triconnected components in the SPQR tree are joined by an edge that represents the virtual edge. See Figure 15.

We treat the SPQR tree of a graph \( G \) as a rooted tree \( T \) by choosing a node \( \nu^* \) as its root. For a node \( \nu \), let \( CH(\nu) \) denote the set of all children of \( \nu \). We denote the graph formed from \( \sigma(\nu) \) by deleting its parent virtual edge \( e' \) as \( \sigma^-(\nu) = (V_\nu, E^-_\nu) \), \((E^-_\nu = E_\nu \setminus \{e'\})\).

Consider the case where \( G \) is a plane graph. When \( G \) is a plane graph, we also treat graphs \( \sigma^-(\nu) \) and \( G^-\nu) \) as plane graphs induced from the embedding of \( G \). A node is called outer if its skeleton has a face whose vertices are all outer vertices of \( G \). We choose the root of SPQR tree \( T \) of \( G \) as an outer node. We always choose an outer R-node as the root of \( T \) (note that \( G \) is assumed to have at least one outer R-node). A real edge in the skeleton of a node is called outer (resp., inner) if it is an outer (resp., inner) edge of \( G \). Then we have the following property (see Appendix D.2 for a proof).

**Lemma 4** Let \( G \) be a biconnected planar graph with a fixed embedding, and \( f \) be a face in \( F \). Then the facial cycle \( f \) is triconnected if and only if the SPQR tree of \( G \) has an R-node \( \nu \) whose skeleton \( \sigma(\nu) \) contains a face \( f' \) with \( V(f') \subseteq V(f) \).
3 Main Theorem

Four points $p_1, p_2, p_3, p_4 \in \mathbb{R}^3$ are called coplanar if they are all in the same plane. Let $P$ be a spherical polyhedron with a fixed position in the $xyz$-coordinate system of $\mathbb{R}^3$. A face $f$ of $P$ is called visible if no point above $f$ is contained in $P$. We place an upward polyhedron $P$ in $\mathbb{R}^3$ so that the bottom face is parallel to the $xy$-plane and $P$ is contained in the upper half of the two spaces separated by the supporting plane of the bottom face, unless stated otherwise. Hence all faces of an upward polyhedron $P$ except for the bottom face are visible. We denote the bottom face of $P$ by $f^a(P)$, and treat $G(P)$ as the plane graph with an outer face $f^a(P)$.

An upward polyhedron $P$ is called proper if
(i) each non-convex face in $P$ contains a separation pair of $G(P)$ in its boundary; and
(ii) for each concave edge $e = (u, v)$ of $P$, there is a non-convex face $f$ of $P$ that contains the both end vertices $u$ and $v$ at the same time ($e$ is not necessarily an edge of the cycle $f$).

Fig. 22 shows examples of proper/non-proper upward polyhedra.

Now we are ready to describe our main theorem, a characterization of upward polyhedral graphs.

Theorem 5 A plane graph $G$ with an outer face $f^a$ is an upward polyhedral graph (i.e. the vertex-edge graph of an upward polyhedron $P$ with the bottom face $f^a$) if and only if it is a biconnected plane graph with $\delta(G) \geq 3$ and $\xi(G) \leq 1$, and the cycle $f^a$ is triconnected in $G$. Moreover, for any such plane graph $G$, there is a proper upward polyhedron $P$ with $G(P) = G$.

Steinitz’ theorem says that separation pairs are the forbidden structure for a planar graph to be realized as convex polyhedra. Theorem 5 tells that separation pairs that attain $\xi(G) \leq 1$ are the forbidden structure for realization of upward polyhedra (Fig. 2 illustrates such separation pairs).

Based on Theorem 5, we can test whether a given plane graph is an upward polyhedral graph or not in linear time, since the splitness $\xi(G)$ and triconnectivity of cycle $f^a$ can be computed in linear time after constructing the SPQR tree in linear time [18, 34].

Theorem 5 implies the sufficiency of Steinitz’ theorem (the necessity is not difficult to show). By Theorem 5, if $G$ is a triconnected plane graph, there is a convex polyhedron $P$ with $G(P) = G$, since a proper upward polyhedron $P$ becomes a convex polyhedron if $G$ has no separation pair.

Fig. 3(a) and (b) show an example of a plane graph $G$ and a proper upward polyhedron $P$ with $G(P) = P$. Fig. 25 shows examples of non upward polyhedral graphs.

Fig. 4(a) shows a straight-line drawing $D$ which is generated from a projected image $I(P)$ of a polyhedron $P$, where the three lines passing through edges $e_1, e_2$ and $e_3$ do not meet at the same point due to a numerical error. However, its underlying graph $G_1$ in Fig. 4(b) is a triconnected planar graph, and Steinitz’ theorem tells that the drawing $D$ retains the topological consistency in the sense that there exists a polyhedron $P$ with $G(P) = G_1$. Fig. 4(c) shows a biconnected planar graph $G_2$ with fixed embedding which admits a spherical polyhedron $G(P) = G_2$. However, Theorem 5 tells that $P$ always has at least two invisible faces from any outer view point, implying that a projected image of such a polyhedron cannot be drawn as any straight-line drawing of $G_2$. Since every facial cycle in $G_2$ is of length at least 5, each face becomes always a star-shaped polygon. Two faces sharing at least two vertices also share an edge, and thereby these faces must be non-coplanar. The SPR tree of $G_2$ has two S-nodes at a P-node, and thus $G_2$ does not satisfy the condition of Theorem 5 for any choice of outer face. Therefore, at least two faces are invisible from any outer view point.

4 Proof for Necessity

This section proves the necessity of Theorem 5. Let $P$ be an upward polyhedron. Obviously $G(P)$ is simple and satisfies $\delta(G(P)) \geq 3$. We see that $G(P)$ is biconnected because otherwise $G(P)$ would have a face whose boundary is not a simple cycle, contradicting the assumption that each facial cycle in a spherical polyhedron is the boundary of a simple polygon (see Fig. 21(a) and (b)).

We then show that $\xi(G(P)) \leq 1$ is necessary (see Appendices D.4 and D.3 for a proof).
Lemma 6 The vertex-edge plane graph \( G(P) \) of an upward polyhedron \( P \) satisfies \( \xi(G(P)) \leq 1 \).

5 Bounding Pyramids

This section prepares a building block of our constructive proof for the sufficiency of Theorem 5.

For three points \( p_1, p_2, p_3 \in \mathbb{R}^3 \), which are not collinear, the plane that contains these points is denoted by \( H(p_1, p_2, p_3) \). A convex polygon with exactly four noncoplanar vertices \( p_1, p_2, p_3, p_4 \in \mathbb{R}^3 \) is called a pyramid, and is denoted by \( \Delta(p_1, p_2, p_3, p_4) \). Consider an ordered sequence of four points \( a, b, c, d \in \mathbb{R}^3 \). The pyramid \( Y = \Delta(a, b, c, d) \) is called of type A if face \( (a, b, c) \) is not visible and face \( (a, b, d) \) is visible. We call the faces \( (a, b, c) \) and \( (a, b, d) \) of a type A pyramid the base face and the top face, respectively. Fig. 7(b) illustrates a pyramid of type A.

The pyramid \( Y = \Delta(a, b, c, d) \) is called of type B if both faces \( (a, b, c) \) and \( (a, b, d) \) are not visible. We call the faces \( (a, b, c) \) and \( (a, b, d) \) of a type B pyramid the side faces. Fig. 8(b) illustrates a pyramid of type B.

The edge \( (a, b) \) of a pyramid of a type A or B is called the splicing edge.

Now we present technical lemmas which prove that any convex polyhedron with two specified adjacent faces can be transformed into the interior of a type A (resp., type B pyramid) pyramid \( Y \) so that two specified faces are contained in the top and the base faces (resp., the side faces) of \( Y \).

Type A Pyramids Let \( P \) be a convex polyhedron, and \( f_0 \) and \( f_1 \) be the two adjacent faces, where we call \( f_0 \) and \( f_1 \) the base face and the top face of \( P \). Let \( e = (u, v) \) be the edge shared by \( f_0 \) and \( f_1 \), which we call the base edge, where we assume that \( u \) and \( v \) appear consecutively in this order when we traverse the top face in the clockwise order (see the base edge \( e = (u, v) \) in Fig. 7(a)).

We say that a convex polyhedron \( P \) with the base face \( f_0 \), the top face \( f_1 \) and the base edge \( e = (u, v) \) fits into a type A pyramid \( Y = \Delta(a, b, c, d) \) in \( \mathbb{R}^3 \) if \( P \) can be placed in \( \mathbb{R}^3 \) so that

1. the positions of vertices \( u \) and \( v \) are equal to \( a \) and \( b \), respectively;
2. the region \( f_0 \) (resp., \( f_1 \)) is contained in the triangle \( (a, b, c) \) (resp., \( (a, b, d) \)); and
3. all faces of \( P \) except for the base face \( f_0 \) are visible.

Fig. 7(c) illustrates a convex polyhedron \( P' \) which fits into the type A pyramid in Fig. 8(c). We show that a triconnected planar graph with the base and top faces can be realized as a convex polyhedron that fits into any type A pyramid (see Appendix D.5 for a proof).

Lemma 7 Let \( G = (V, E, F) \) be a triconnected planar graph, and \( Y = \Delta(a, b, c, d) \) be a type A pyramid in \( \mathbb{R}^3 \). Then for any two adjacent faces \( f_0 \) and \( f_1 \) in \( G \) designated as the base and the top faces, there is a convex polyhedron \( P \) with \( G(P) = G \) that fits into \( Y \).

Type B Pyramids Let \( P \) be a convex polyhedron, and \( f_{s1} \) and \( f_{s2} \) be the two adjacent faces, where we call \( f_{s1} \) and \( f_{s2} \), the side faces of \( P \). Let \( e = (u, v) \) be the edge shared by \( f_{s1} \) and \( f_{s2} \), which we call the base edge (see the base edge \( e = (u, v) \) in Fig. 8(a)).

We say that a convex polyhedron \( P \) with the two side faces \( f_{s1} \) and \( f_{s2} \) and the base edge \( e = (u, v) \) fits into a type B pyramid \( Y = \Delta(a, b, c, d) \) in \( \mathbb{R}^3 \) if \( P \) can be placed in \( \mathbb{R}^3 \) so that

1. the positions of vertices \( u \) and \( v \) are equal to \( a \) and \( b \), respectively;
2. the region \( f_{s1} \) (resp., \( f_{s2} \)) is contained in the triangle \( (a, b, c) \) (resp., \( (a, b, d) \)); and
3. all faces of \( P \) except for the side faces \( f_{s1} \) and \( f_{s2} \) are visible.

Fig. 8(c) illustrates a convex polyhedron \( P' \) which fits into the type B pyramid in Fig. 8(b).

Any triconnected planar graph with the side faces can be realized as a convex polyhedron that fits into a given type B pyramid (see Appendix D.5 for a proof).

Lemma 8 Let \( G = (V, E, F) \) be a triconnected planar graph, and \( Y = \Delta(a, b, c, d) \) be a type B pyramid in \( \mathbb{R}^3 \). Then for any two adjacent faces \( f_{s1} \) and \( f_{s2} \) in \( G \) designated as the two side faces, there is a convex polyhedron \( P \) with \( G(P) = G \) that fits into \( Y \).
6 Proof for Sufficiency

We prove the sufficiency of Theorem 5 by presenting an algorithm that constructs a desired upward polyhedron for a given plane graph $G$ which satisfies the necessary condition in Theorem 5.

We first present the main idea of the algorithm. The SPQR tree decomposition of the given graph $G$ tells us the connection among the triconnected components, where the S- and P-nodes represent how the R-nodes are joined each other. Recall that the structure of an R-node is given by its skeleton $\sigma(\nu^s)$, which is a triconnected planar graph and admits a convex polyhedron $P_\nu$ with $G(P_\nu) = \sigma(\nu)$ by Steinitz’ theorem. In fact, our upward polyhedron $P$ consists of these convex polyhedra $P_\nu$, and for the necessary projective transformations using bounding pyramids. Of course, any biconnected plane graph $G$ can have the SPQR tree decomposition, but in general, it is impossible to combine the convex polyhedra $P_\nu$ for all R-nodes into a single spherical polyhedron, as observed in Lemma 3. However, if $\kappa(G) \geq 2$, $\delta(G) \geq 3$ and $\xi(G) \leq 1$ hold and the outer face $f^o$ is triconnected, then we show that there exists an algorithm that combines those convex polyhedra $P_\nu$ into a proper upward polyhedron $P$, which will prove the sufficiency of Theorem 5.

We next give an outline of the algorithm. Choose an R-node $\nu^s$ whose skeleton contains at least three outer vertices of $G$, and designate $\nu^s$ as the root of the SPQR tree of $G$. We first realize the skeleton $\sigma(\nu^s)$ of the root R-node $\nu^s$ as an upward convex polyhedron $P_{\nu^s}$ of the R-node as follows. Fig. 9 shows an upward convex polyhedron $P = P_{\nu}$ with $G(P_{\nu}) = \sigma(\nu)$ for the root R-node $\nu^s$ in the SPQR tree of the graph $G$ in Fig. 3(a). We choose a type A pyramid $Y = (a, b, c, d)$ on the $xy$-plane ($H(a, b, c)$ is the $xy$-plane and $d$ is a point over the $xy$-plane). By designating an outer edge $e = (u, v)$ of $\sigma(\nu^s)$ as the base edge, and the outer face $f^o(\sigma(\nu^s))$ and the face $f$ sharing $e$ as the base and top faces, respectively, we convert $P_{\nu^s}$ into a convex polyhedron $P$ that fits into $Y$. By Lemma 7, the resulting polyhedron $P$ is upward.

Starting with setting $P$ as an upward convex polyhedron $P_{\nu^s}$ for the root R-node $\nu^s$, the algorithm traverses the rooted SPQR tree in a DFS manner, and computes an “appropriate” convex polyhedron $P_\nu$ whenever it visits an R-node $\nu$. The polyhedron $P_\nu$ will be combined with the current polyhedron $P$ to update $P$ as a new upward polyhedron. We construct a type A or type B pyramid $Y$ on a face of $P$ to choose the “appropriate” convex polyhedron for $P_\nu$ (the details will be described in Section 6.2). Then the convex polyhedron $P_\nu$ will be attached to the current polyhedron $P$ in such a way that (i) the bottom face $f^o(P_\nu)$ is completely contained in the region of some face $f$ of $P$, and (ii) one edge $e$ of $f^o(P_\nu)$ is completely contained in some edge of $f$ without generating any other intersection between $f^o(P_\nu)$ and $f$. This means that all faces in $P$ are always simple polygons.

After investigating the structure of S- and P-nodes in the next subsection, we describe procedures for S-, R- and P-nodes. Fig. 16(1)-(9) illustrates a computational process of our algorithm applied to the graph in Fig. 3(a).

6.1 Types of nodes in SPQR trees and virtual edges

For a biconnected plane graph $G$ with $\delta(G) \geq 3$ and $\xi(G) \leq 1$ which has a triconnected outer face $f^o$, the root of the SPQR tree of $G$ is now chosen as an R-node $\nu^s$, whose skeleton contains at least three outer vertices of $G$. For each non-root R-node $\nu$, we treat the skeletons $\sigma^-(\nu)$ and $\sigma(\nu)$ as plane graphs induced from the plane embedding of $G$. Note that $\sigma(\nu)$ contains the virtual parent edge of $\nu$, and is triconnected. For an edge $(u, v)$, we denote by the two faces of $\sigma(\nu)$ containing an edge $(u, v)$ by $f^u_{\nu^o}$ and $f^v_{\nu^o}$, where $u$ and $v$ appear in this order when we traverse the cycle $f^o_{\nu^o}$ in the clockwise order. (see Fig 8(c)). Then a biconnected simple plane graph $G$ with $\delta(G) \geq 3$ and $\xi(G) \leq 1$ is characterized by the following conditions:

(C1) Each outer P-node $\nu$ has no inner S-node child and no inner real edge in its skeleton.

(C2) Each non-outer P-node $\nu$ with an S-node child has no other S-node child and no real edge in its skeleton ($\nu$ cannot have an outer S-node or an outer real edge).

(C3) For each S-node, no two real edges are adjacent in the skeleton $\sigma^-(\nu)$. 
By condition (C1), for an outer P-node $\nu$, exactly one edge in $\sigma^-(\nu)$ corresponds to an outer R-node, an outer S-node or an outer real edge, and the rest of edges correspond to only R-nodes (note that the virtual parent edge of $\sigma(\nu)$ may be an outer real edge or correspond to an outer S-node). We call an outer P-node $\nu$ type R (resp., type S and type E) if all edges in $\sigma^-(\nu)$ correspond to R-nodes (resp., one edge in $\sigma^-(\nu)$ corresponds to an outer S-node, and one edge in $\sigma^-(\nu)$ is an outer real edge).

By condition (C2), a non-outer P-node $\nu$ can have at most one of a non-outer S-node or an inner real edge. We call a non-outer P-node $\nu$ type R (resp., type S and type E) if all edges in $\sigma^-(\nu)$ correspond to R-nodes (resp., one edge in $\sigma^-(\nu)$ corresponds to an S-node, and one edge in $\sigma^-(\nu)$ is a real edge).

We distinguish the following three cases for a virtual edge $e$ in the current polyhedron $P$: (i) $e$ is not in the boundary of the bottom face $f^o(P)$ of $P$, and $e$ is a convex edge; (ii) $e$ is not in the boundary of $f^o(P)$, and $e$ is a concave edge; and (iii) $e$ is in the boundary of $f^o(P)$ (see Fig. 10). We always treat such an edge $e$ in (iii) as a concave edge which is created by the side face containing $e$ and the plane $H(f^o(P))$, because we are not allowed to generate any polyhedron under the plane $H(f^o(P))$.

6.2 Replacing virtual edges

This subsection describes how to process virtual edges in the current upward polyhedron $P$. In the following, we omit the details on the treatment of visibility of non-bottom faces, and star-shaped faces for simplicity of algorithm description. We focus on how to add a new polyhedron $P_\nu$ to the current polyhedron $P$ without losing the upward property.

Let $e = (u, v)$ be the virtual edge to be processed (i.e., the virtual edge with the highest priority in the DFS traversal of the SPQR tree). The virtual edge $e = (u, v)$ corresponds to an S-, R-, or P-node. Let $f_1, f'_1$ be the two faces of $P$ that share the edge $e = (u, v)$, where we assume that $f'_1$ denotes the region obtained from the plane $H(f^o(P))$ by removing the region $f^o(P)$ if $e$ is in the boundary of $f^o(P)$. Also assume that $u$ and $v$ appear in this order when we traverse the facial cycle $f_1$ in the clockwise order (see Fig. 10). Let $\nu$ be the node to which $e$ corresponds. We now distinguish the three cases: $\nu$ is an S-node, R-node, or P-node.

**S-node case:** First we consider the case where $e = (u, v)$ corresponds to an S-node $\nu$. In this case, we just invoke procedures for R-nodes or P-nodes, which are described below. We number the child edges of $\nu$ as $e_1, e_2, \ldots, e_{|E_{\nu}^-|}$ by traversing these edges from $u$ to $v$, and we denote this by $E_{\nu}^- = (e_1, e_2, \ldots, e_k)$, where $k = |E_{\nu}^-|$. Let $\mu_i \in Ch(\nu)$ denote the child node corresponding to $e_i$. We first partition the line segment $[u, v]$ for the edge $e = (u, v)$ in $P$ into $k$ line segments $[u_i, u_{i+1}]$, $i = 1, 2, \ldots, k$, where $u_1 = u$ and $u_{k+1} = v$. From $i = 1$ to $k$, assign $e_i$ to the $i$-th segment $[u_i, u_{i+1}]$, and invoke the procedure for R-nodes (resp., P-nodes) if $e_i$ corresponds to an R-node (resp., a P-node). If $e_i$ is a real edge, then the $i$-th segment $[u_i, u_{i+1}]$ realizes the edge in the resulting polyhedron.

**R-node case:** Next we consider the case where $e = (u, v)$ corresponds to an R-node $\nu$. The treatment of R-nodes is the most essential in our construction. We first construct a convex polyhedron $P_\nu$ of the skeleton $\sigma(\nu)$. We remark that $\sigma(\nu)$ contains the parent virtual edge $e = (u, v)$, which is not an element of a polyhedron for the input graph $G$, and thus we have to eliminate it when we add $P_\nu$ to the current polyhedron $P$.

1. We first consider the case when $e = (u, v)$ is a convex edge in $P$. Let us designate $e = (u, v)$ as the base edge, and $f_{\nu}^{u+}, f_{\nu}^{v-}$ as the base and top faces of $P_\nu$. We then choose a type $A$ pyramid $Y = \Delta(a, b, c, d)$ so that $a$ and $b$ are the endpoints of $(u, v)$ in $P$, $c$ is a point in the face $f_1$, and $d$ is a point in the plane $H(f_1)$ and above the face $f_1$ (note that such $d$ can be chosen so that $\Delta(a, b, c, d)$ is type $A$, since $P$ is upward and $e$ is convex). By Lemma 7, we can assume that $P_\nu$ fits into $Y$. By definition of fitting polyhedra, we see that the polyhedron $P$ modified by adding $P_\nu$ remains upward, and the parent virtual edge $e$ temporarily introduced to construct $P_\nu$ has disappeared in the resulting $P$, since the top face $f_{\nu}^{v-}$ of $P_\nu$ is contained in the plane $H(f_1)$ (see Fig. 7(f)). Note that
by choosing a point $c$ outside $H(f'_1)$, we can realize edge $e$ as an element of the resulting polyhedron, if necessary. We use this technique when we handle $P$-nodes.

(2) We next consider the case when $e = (u, v)$ is a concave edge in $P$ (including the case where $e$ is in the boundary of $f''(P)$). Let us designate $e = (u, v)$ as the base edge, and $f''_w, f''_v$ as the side faces of $P$. We then choose a type B pyramid $Y = \Delta(a, b, c, d)$ so that $a$ and $b$ are the endpoints of $(u, v)$ in $P$, $c$ is a point in the face $f_1$, and $d$ is a point in the face $f'_1$ (note that such $c, d$ can be chosen so that $\Delta(a, b, c, d)$ is type B, since $P$ is upward and $e$ is concave). By Lemma 8, we can assume that $P$ fits $Y$. By definition of fitting polyhedra, we see that the polyhedron $P$ modified by adding $P$ remains upward, and the edge $e$ in $P$ is concealed by the two invisible side faces of $P$ in the resulting $P$ (see Fig. 8(f)).

**P-node case:** Finally we consider the case where $e = (u, v)$ corresponds to a P-node $\nu$. For the parent cut-pair $\{u, v\}$ of a P-node $\nu$, we number the child edges of $\nu$ as $e_1, e_2, \ldots, e_{|E^-\nu|}$ by traversing these edges from left to right in the plane graph $\sigma^-(\nu)$. We denote this by $E^-\nu = (e_1, e_2, \ldots, e_j, \ldots, e_k)$, where $k = |E^-\nu|$, and $e_j$ denotes the real edge or corresponds to a child $S$-node (if any). We assume $1 \neq j^*$ without loss of generality. When $e = (u, v)$ is in the boundary of $f''(P)$, i.e., $\nu$ is an outer P-node, it holds that $j^* = k$ by condition (C1) of $G$.

Let $\mu_i \in Ch(\nu)$ denote the child node corresponding to $e_i$ (see Figs. 11(a), 12(b) and (c)). In what follows, for simplicity, we discuss the concave edge case including the case when $e$ is in the boundary of $f''(P)$, for which no polyhedron will be placed on the region $f'_1 \subset H(f''(P))$ due to condition $j^* = k$.

(i) **type R.** Let $\nu$ be a type R P-node, where all child nodes $\mu_i$ are R-nodes (see Fig. 11(a)).

(1) First we consider the case when $e = (u, v)$ is a convex edge in $P$. For each $i = 1, 2, \ldots, k$, let us designate $e = (u, v)$, and $f''_{\mu_i}, f''_{\mu_i}$ as the base edge and the base and top faces of $P$, and prepare a type A pyramid $Y_i$ as follows. We choose $k + 1$ distinct planes $H_1, H_2, \ldots, H_{k+1}$ passing through the edge $e$ of $P$, where $H_i = H(f_i), H_{k+1} = H(f'_1)$, and the half planes $H'_i, H'_2, \ldots, H'_{k+1}$ of $H_1, H_2, \ldots, H_{k+1}$ in the exterior of $P$ appear in this order around $e$. Let $Y_i = \Delta(a, b, c_i, d_i)$, $i = 1, 2, \ldots, k$, be a pyramid of type A such that $a$ and $b$ are the endpoints of $(u, v)$ in $P$, $c_i$ is a point in the top face of $f'_{i}$ (or $f'_1$ for $i = 1$), and $d_i$ is a point in the plane $H'_i$ (see Fig. 11(b)). Analogously with the R-node case, we choose an upward convex polyhedron $P_{\mu_i}$ that fits into $Y_i$ for each $i$. Then we stack $P_{\mu_1}, \ldots, P_{\mu_k}$ placing each on the previous one. More specifically, we choose the position of $c_i$ of the $i$-th pyramid $Y_i$ after deciding the polyhedron $P_{\mu_i}$ for $\sigma(\mu_i)$: $c_i$ will be chosen as an internal point of the region of the top face $f''_{\mu_i}$ of $P_{\mu_i}$. Hence we need to process child nodes $\mu_i, \ldots, \mu_k$ in this order. By the choice of half planes $H'_1, \ldots, H'_{k+1}$, the edge will disappear after adding the last polyhedron $P_{\mu_k}$, and we easily see that the augmented polyhedron remains upward.

(2) For the case where $e$ is concave, we briefly describe how to augment the current polyhedron $P$. In a similar manner, we choose half planes $H'_1, \ldots, H'_{k+1}$ between $f_1$ and $f'_1$, where the difference from the convex case is that $H'_k$ is chosen so that the edge $e$ shared by $H'_k$ and $f'_1$ remains visible. Analogously with the convex case, we prepare type A pyramids $Y_1, \ldots, Y_{k-1}$ for the R-nodes $\mu_1, \ldots, \mu_{k-1}$, prepare their convex polyhedra $P_{\mu_1}, \ldots, P_{\mu_{k-1}}$ that fit into $Y_1, \ldots, Y_{k-1}$, and stack these polyhedra on the face $f_1$. See Fig. 11(c). By the choice of $H'_k$, the edge $e$ is visible and concave in the resulting polyhedron $P$. For the last R-node child $\mu_k$, we use the procedure for virtual concave edges corresponding R-node. Thus, we prepare a type B pyramid $Y_k$ and place an upward polyhedron $P_{\mu_k}$ that fits into $Y_k$ on $P$.

(ii) **type E.** Let $\nu$ be a type E P-node, where edge $e_j \in E^-(\nu)$ is a real edge, and all the other edges in $E^-(\nu)$ correspond to R-nodes (see Fig. 12(a)).

(1) We first consider the case when $e = (u, v)$ is a convex edge in $P$. We first stack upward
polyhedra $P_{\mu_1}, \ldots, P_{\mu_{j_*}}$ for skeletons $\sigma(\mu_1), \ldots, \sigma(\mu_{j_*})$, using the exactly same method as type R P-nodes. We apply the same procedure to the remaining R-nodes $P_{\mu_{j_*+1}}, \ldots, P_{\mu_k}$ by exchanging the roles of $f_1$ and $f'_1$.

(2) We next consider the case when $e = (u, v)$ is a concave edge in $P$. We can treat this case using the same method as the convex edge case, where we only need to choose the half planes for the top faces of $P_{\mu_{j_*}}$ and $P_{\mu_{j_*+1}}$ so that the edge $e$ remains visible.

(iii) type S. Let $v$ be a type S P-node, where $\mu_{j_*}$ is an S-node, and all the other children $\mu_i$ are R-nodes (see Fig. 12(b)). This can be treated as the case of type S P-nodes, followed by the procedure for the case of S-nodes. We first stack polyhedra $P_{\mu_1}, \ldots, P_{\mu_{j_*}}$ and $P_{\mu_{j_*+1}}, \ldots, P_{\mu_k}$ in the same manner as the case of type S P-nodes. Then we treat the edge $e$ as an S-node, i.e., apply the procedure for S-nodes. This completes the description of the algorithm.

We now briefly show the correctness of our algorithm. As we observed above, the algorithm realizes real edges in R-, S- and P-nodes in the current polyhedron $P$, and the set of edges of the final polyhedron consists only of the real edges of $G$, i.e., $G(P) = G$. By the way of choosing convex polyhedra $P_0$ that fit into bounding pyramids on the current polyhedron $P$, all non-bottom faces of $P$ are always visible. Also, when we place a convex polyhedra $P_0$, we keep the convexity of the base edge $e$ if $e$ was convex in $P$ before adding $P_0$ to $P$. This ensures that the resulting polyhedron $P$ remains proper, since a concave edge can appear only along the boundary of some $f^0(P_0)$ for the first time, and its endpoints remain on the boundary of the face $f$ on which $P_0$ is placed. We can also easily maintain all the faces as star-shaped as follows. When we introduce a new face $f$ in the current polyhedron $P$ (i.e., $f$ is a non-bottom face of $P_0$), we choose an internal point in the region of $f$ as the view point $r_f$ of $f$. When the face $f$ will be modified by placing the bottom face $f^0(P_0)$ of a polyhedron $P_0$ on $f$ (or $f$ will be extended with a face $f_1$ of a polyhedron $P_0$), we choose the polygonal shape of such face $f^0(P_0)$ (or $f_1$) so that the view point $r_f$ remains as a visible point in the resulting face $f$.

We can transform a convex polyhedron with $n$ vertices into a convex polyhedron that fits into a pyramid $Y$ in $O(n)$ time. Hence our algorithm for realizing an upward polyhedron of a given graph $G = (V, E, F)$ can be implemented to run in $O(n + T(n))$ time, where $T(n)$ denotes the time complexity of any algorithm for constructing a convex polyhedron of a triconnected planar graph with $n$ vertices. The current best algorithm runs in $O(n^{1.5})$ time [9, 21].
References

Figure 1: Illustrations of crown-shaped polyhedra supported by the $xy$-plane: (a) a crown-shaped polyhedron with a view point below its supporting plane; (b) a crown-shaped polyhedron with a view point above its supporting plane; (c) an upward-visible polyhedron.
Figure 2: (a) and (b) show a pair of outer vertices $u$ and $v$ with $\xi(u, v) = 1$, where two inner faces $f_1, f_2$ and the outer face $f^o$ are joined at $\{u, v\}$, and $f_1$ and $f_2$ are linearly-joined at $\{u, v\}$; (c) and (d) show a pair of vertices $u$ and $v$ with $\xi(u, v) = 2$, where inner faces $f_1, f_2, f_3$ and $f_4$ are joined at $\{u, v\}$, and $f_i$ and $f_{i+1}$ ($i = 1, 3$) are linearly-joined at $\{u, v\}$ ($f_2 \neq f_1 \neq f_4 \neq f_3$ but possibly $f_2 = f_3$).

Figure 3: (a) A biconnected plane graph $G$ which satisfies the necessary condition of Theorem 5: (b) a proper upward polyhedron $P$ with $G(P) = G$ for the graph $G$ in (a), where the position of vertices $v_1, v_2, v_3$ and $v_4$ are the same as those in the polyhedron $P_0$ in Fig. 9(b).

Figure 4: (a) A straight-line drawing $D$; (b) the underlying graph $G_1$ of the straight-line drawing in (a); (c) a planar graph $G_2$ with fixed embedding, where for any spherical polyhedron $G(P) = G_2$, at least two faces are invisible from any viewpoint outside $P$. 
Figure 5: Contradictory straight-line plane drawings $I_P$ of $G(P)$ of upward polyhedra $P$.

Figure 6: (a), (b) Contradictory straight-line plane drawings $I_P$ of $G(P)$ of upward polyhedra $P$; (c) a biconnected plane graph $G$ whose outer facial cycle is not triconnected; i.e., no R-node $\nu$ with three outer vertices of $G$ in its skeleton $\sigma(\nu)$. 

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Figure 7: (a) A convex polyhedron $P$ with base face $(u, v, v_1, v_2)$ and top face $(u, v, v_4, v_3)$; (b) a pyramid $\Delta(a, b, c, d)$ of type A with base face $(a, b, c)$ and top face $(a, b, d)$; (c) a convex polyhedron $P'$ transformed from $P$ into the interior of pyramid $\Delta(a, b, c, d)$ of (b); (d) the skeleton $\sigma(v)$ of an R-node $v$; (e) a type A pyramid $Y$; (f) an upward polyhedron $P'_v$ that fits into $Y$.

Figure 8: (a) A convex polyhedron $P$ with side faces $(u, v, v_1, v_2)$ and $(u, v, v_4, v_3)$; (b) a pyramid $\Delta(a, b, c, d)$ of type B with side faces $(a, b, c)$ and $(a, b, d)$; (c) a convex polyhedron $P''$ transformed from $P$ into the interior of pyramid $\Delta(a, b, c, d)$ of (b); (d) the skeleton $\sigma(v)$ of an R-node $v$; (e) a type B pyramid $Y$; (f) a convex polyhedron $P'_v$ that fits into $Y$. 

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Figure 9: (a) The skeleton $\sigma(v^\ast)$ of the root R-node $v^\ast$ of the SPQR tree in Fig. 15; (b) an upward convex polyhedron $P_{o^\ast}$ with $G(P_{o^\ast}) = \sigma(v^\ast)$.

Figure 10: (a) A convex virtual edge $e$ not on the boundary of the bottom face $f^\alpha(P)$; (b) a concave virtual edge $e$ not on the boundary of the bottom face $f^\alpha(P)$; (c) a virtual edge $e$ on the boundary of the bottom face $f^\alpha(P)$.

Figure 11: (a) The skeleton $\sigma(v)$ of a type R P-node $v$; (b) type A pyramids $Y_1, Y_2, \ldots, Y_k$ for convex edge $e$; (c) type A pyramids $Y_1, Y_2, \ldots, Y_k$ and a type B pyramid $Y_{k+1}$ for concave edge $e$. 
Figure 12: (a) The skeleton $\sigma(v)$ of a type E P-node $v$; (b) the skeleton $\sigma(v)$ of a type S P-node $v$; (c) type A pyramids $Y_1, Y_2, \ldots, Y_k$ for convex edge $e$; (d) type A pyramids $Y_1, Y_2, \ldots, Y_k$ for concave edge $e$. 
A Concave Corner Constraint

We show that characterizing general spherical polyhedra is as hard as characterizing spherical polyhedra with the concave corner constraint. Thus, a graph-theoretical characterization of spherical graphs can tell us when a given biconnected plane graphs with a set $A$ of specified corners admits a spherical polyhedron $P$ with $G(P) = G$ such that all the corners in $A$ appear as concave corners in $P$. To show this, we augment $G$ with $A$ to a biconnected plane graph $G'$ as follows. For each corner $c$, we choose an edge $e = (v_i, v_j)$ in the facial cycle $f$, insert $K_4$ on new vertices $a_i, b_i, c_i, d_i$ with edge $(a_i, c_i)$ subdivided with the vertex $v_i$ so that the edge $e$ ends with $a_i$, as shown in Fig. 13(b). Define $G'$ be the graph obtained by augmenting each corner in $A$ with such a subgraph.

![Figure 13: (a) A biconnected plane graph $G$, where the corner of face $f$ at vertex $v_i$ belongs to $A$; (b) A biconnected plane graph $G'$ obtained by augmenting each corner in $A$ with a subgraph.](image)

**Theorem 9** Let $G = (V, E, F)$ be a simple biconnected planar graph with a fixed embedding, and let $A$ be a set of corners. Then $G$ is the vertex-edge graph of a spherical polyhedron such that all the corners in $A$ appear as concave corners if and only if the graph $G'$ defined in the above is the vertex-edge graph of a spherical polyhedron.

**Proof.** Let $A' = \{(v_i, f, e) \mid (v_i, f) \in A\}$ be the set of tuples $(v_i, f, e)$ of a corner $(v_i, f)$ and an edge $e$ for which we inserted $K_4$ to obtain $G'$.

We first show the only-if part. Let $P$ be a spherical polyhedron with $G = G(P)$ such that all the corners in $A$ appear as concave corners (see Fig. 14(a)). Then a spherical polyhedron $P'$ with $G_3 = G(P')$ can be constructed from $P$ by attaching a pyramid $\Delta$ for each $(v_i, f, e) \in A'$. More formally, we attach a pyramid $\Delta$ so that one of the side is split into the line segment $[a_i, v_i]$ along the line segment $[v_i, v_i]$ and the line segment $[v_i, v_i]$ inside the polygon for the face $f$ (see Fig. 14(b)).

We next show the if part. Let $P'$ be a spherical polyhedron with $G_3 = G(P')$. For each $(v_i, f, e) \in A'$, we first show that the line segments $[v_i, a_i]$, $[v_i, c_i]$ and $[v_i, c_i]$ appear in this order along a straight-line in $P'$. Let $f'$ be the other face of $G'$ containing edge $e$ (see Fig. 13(b)), and $f_i$ be the face containing edges $e_i$ and $(v_i, c_i)$. Since $f$ and $f'$ share edge $e = (v_i, a_i)$ and vertex $v_i$, the line segments $[v_i, a_i]$ and $[a_i, v_i]$ must lie on a straight-line in $P'$. Similarly, since $f$ and $f_i$ share edge $(v_i, c_i)$ and vertex $v_i$, the line segments $[v_i, c_i]$ and $[a_i, v_i]$ must lie on a straight-line in $P'$. Therefore, the line segments $[v_i, a_i]$, $[a_i, v_i]$, $[v_i, c_i]$ must appear in this order along a straight-line in $P'$.

Let $(v_i, v_k)$ be the edge next to $e$ along face $f$. Then to prove the if part, it suffices to show that the angle made by line segments $[a_i, v_i]$ and $[v_i, v_k]$ inside the region for $f$ (ignoring the inserted $K_4$ at $v_i$) is greater than $\pi$ (see Fig. 14(b)). For this, we first show that the corner of $f$ at vertex $a_i$ is convex. Assume that the corner is concave, as shown in Fig. 14(c). Then the interior of the face $f$ intersects with line segment $[a_i, v_i]$. However, edge $(a_i, v_i)$ is realized as line segment $[a_i, v_i]$ in $P'$, and this means that the polygon for $f$ and line segment for edge $(a_i, v_i)$ are not disjoint, contradicting that $P'$ is a spherical polyhedron. Hence the corner of $f$ at vertex $a_i$ is convex (see Fig. 14(c)). Assume indirectly that the angle made by line segments $[a_i, v_i]$ and $[v_i, v_k]$ inside the region for $f$ (ignoring the inserted $K_4$ at $v_i$) is less than or equal to $\pi$. In this case, line segment $[v_i, v_k]$ intersects with line segment $[v_i, c_i]$ or the triangle $[a_i, b_i, c_i]$. The former case contradicts the disjointness of line segments for edges in a spherical polyhedron. The latter case implies that the path from $v_i$ to $a_i$ along facial cycle $f$ intersects with the boundary of the triangle $[a_i, b_i, c_i]$, which again contradicts the disjointness of line segments for edges in a spherical polyhedron.
Figure 14: (a) Illustration of a spherical polyhedron $P$ with $G(P) = G$ for the graph $G$ in Fig 13(a); (b) Illustration of a spherical polyhedron $P'$ with $G(P') = G'$ for the graph $G'$ in Fig 13(b); (c) The case where the inner angle at $a_i$ is greater than $\pi$. (d) The case where he angle made by line segments $[a_i, v_i]$ and $[v_i, v_k]$ is less than or equal to $\pi$.

B SPQR Decompositions

We review the definition of triconnected components [34] (or 3-blocks) and a variation of the SPQR tree [18, 19] of a biconnected graph.

First we review the definition of triconnected components [34]. If $G$ is triconnected, then $G$ itself is the unique triconnected component of $G$. Otherwise, let $u, v$ be a cut-pair of $G$. We split the edges of $G$ into two disjoint subsets $E_1$ and $E_2$, such that $|E_1| > 1$, $|E_2| > 1$, and the subgraphs $G_1$ and $G_2$ induced by $E_1$ and $E_2$ only have vertices $u$ and $v$ in common. Form the graph $G'_1$ from $G_1$ by adding an edge (called a virtual edge) between $u$ and $v$ that represents the existence of the other subgraph $G_2$. Similarly form $G'_2$. We continue the splitting process recursively on $G'_1$ and $G'_2$. The process stops when each resulting graph reaches one of three forms: a triconnected simple graph, a set of three multiple edges (a triple bond), or a cycle of length three (a triangle).

The triconnected components of $G$ are obtained from these resulting graphs:

- a triconnected simple graph;
- a bond, formed by merging the triple bonds into a maximal set of multiple edges;
- a polygon, formed by merging the triangles into a maximal simple cycle.

The triconnected components of $G$ are unique. See [34] for further details.

One can define a tree structure, sometimes called as the 3-block tree, using triconnected components as follows. The nodes of the 3-block tree are the triconnected components of $G$. The edges of the 3-block tree are defined by the virtual edges, that is, if two triconnected components have a virtual edge in common, then the nodes that represent the two triconnected components in the 3-block tree are joined by an edge that represents the virtual edge.

There are many variants of the 3-block tree in the literature; the first was defined by Tutte [63]. In this paper, we use the terminology of the SPQR tree, as defined by di Battista and Tamassia [18, 19]. We now briefly review this terminology.

Each node $\nu$ in the SPQR tree is associated with a graph $G = (V, E)$ called the skeleton of $\nu$, denoted by $\sigma(\nu) = (V_\nu, E_\nu)$ ($V_\nu \subseteq V$), which corresponds to a triconnected component. There are four types of nodes in...
the SPQR tree. The node types and their skeletons are:

1. Q-node: the skeleton consists of two vertices connected by two multiple edges. Each Q-node corresponds to an edge of the original graph.
2. S-node: the skeleton is a simple cycle with at least 3 vertices (this corresponds to a polygon triconnected component).
3. P-node: the skeleton consists of two vertices connected by at least 3 edges (this corresponds to a bond triconnected component).
4. R-node: the skeleton is a triconnected graph with at least 4 vertices.

The SPQR tree as developed by di Battista and Tamassia is a data structure with efficient operations. In this paper, we use the SPQR tree only as a convenient way to traverse the 3-blocks of a biconnected graph. In fact, we use a slight modification of the SPQR tree: we omit the Q-nodes and we root the tree as described below. We will refer the (modified) SPQR tree as the SPQR tree throughout this paper.

The SPQR tree is unique [18, 19]. We treat the SPQR tree of a graph $G$ as a rooted tree $T$ by choosing an arbitrary node $v^*$ as its root.

Figure 15 shows the rooted SPQR tree of the biconnected plane graph $G$ in Fig. 3(a), which can be divided into triconnected components as the nodes in Figure 15. Here, $v^*$, $v_b$, $v_c$, $v_d$, $v_f$, $v_h$, $v_j$, $v_k$, and $v_m$ are R-nodes, $v_a$, and $v_l$ are S-nodes, and $v_d$, $v_j$, and $v_l$ are P-nodes.

![](image)

Figure 15: The SPQR tree of the graph $G$ in Fig. 3(a).

Some further notation for the SPQR tree is required. Suppose that $G$ is a biconnected (but not triconnected) planar graph, and $T$ is the rooted SPQR tree of $G$. Let $v$ be a non-root node in $T$, and $\mu$ be the parent of $v$. The graph $\sigma(\mu)$ has one virtual edge $e$ in common with $\sigma(v)$. The edge $e$ is the parent virtual edge in $\sigma(v)$, and it is a child virtual edge in $\sigma(\mu)$. The parent virtual edge of a node $v$ is denoted by $\text{parent}(v)$ (we let $\text{parent}(v) = \emptyset$ if $v$ is the root). We define a parent cut-pair of $v$ as the two endvertices of a parent virtual edge $e$. For each non-root node $v$ of the SPQR tree, $\sigma(v)$ has precisely one parent virtual edge, and for each non-leaf node $v'$, $\sigma(v')$ has at least one child virtual edge. For a node $v$, let $Ch(v)$ denote the set of all children of $v$. Note that each child virtual edge in $\sigma(v)$ corresponds to a child in $Ch(v)$.

We denote the graph formed from $\sigma(v)$ by deleting its parent virtual edge as $\sigma^-(v)$, if $v$ is not the root of $T$.

### C A computational process for the graph in Fig. 3(a)

Fig. 16 illustrates a computational process of the algorithm applied to the graph in Fig. 3(a).
Figure 16: A computational process for the graph in Fig. 3(a): (1) The virtual edge \((v_2, v_3)\) in \(P_1 = P_{v_a}\) corresponding to the S-node \(v_a\) is replaced with the real edges \((v_2, v_{10})\) and \((v_9, v_6)\) and the virtual edge \((v_{10}, v_9)\), corresponding to the R-node \(v_b\); (2) after attaching a convex polyhedron \(P_{v_a}\) to \(P_1 = P_{v_a}\), the virtual edge \((v_2, v_3)\) corresponding to the R-node \(v_c\) is replaced with a convex polyhedron \(P_{v_c}\); (3) the virtual edge \((v_1, v_2)\) corresponding to the P-node \(v_d\) is processed by attaching a convex polyhedron \(P_{v_d}\) for its child R-node \(v_c\) realizing the edge \((v_1, v_2)\) as the real edge; (4),(5) The virtual edge \((v_5, v_7)\) corresponding to the P-node \(v_f\) is processed by attaching a convex polyhedron \(P_{v_f}\) for its first child R-node \(v_g\) and a convex polyhedron \(P_{v_h}\) for its second child R-node \(v_i\); (6),(7) The virtual edge \((v_5, v_6)\) corresponding to the P-node \(v_i\) is processed by attaching a convex polyhedron \(P_{v_k}\) for its first child R-node \(v_k\) and a convex polyhedron \(P_{v_{28}}\) for its last child R-node \(v_j\); (8) The child S-node \(v_l\) of \(v_i\) is replaced with a convex polyhedron \(P_{v_{28}}\) for its child R-node \(v_m\) and the real edge \((v_5, v_{28})\).
D Proofs of Lemmas

D.1 Proof of Lemma 3

Let $G$ be a biconnected planar graph with a fixed embedding, and let $G$ have a separation pair $\{u, v\}$ with $\xi(u, v) \geq 3$ which has three edges $e_i = (u, w_i), \ i = 1, 2, 3$ incident to $u$ such that each edge $e_i = (u, w_i)$ is shared by a pair of linearly-joined faces $f_i$ and $f'_i$ at $\{u, v\}$. Assume that a spherical polyhedron $P$ with $G(P) = P$ exists. Since $f_i$ and $f'_i$ share the edge $(u, w_i)$ (and $v$), the faces $f_i$ and $f'_i$ of $P$ must be contained in different planes in $\mathbb{R}^2$. This implies that the edge $(u, w_i)$ and $v$ lie on the same straight-line $L_i$ in $\mathbb{R}^2$. However, two distinct points $u$ and $v$ lie on $L_i$ and we have $L_1 = L_2 = L_3$. Hence, $e_1, e_2$ and $e_3$ lie on the same straight-line. However, these three edges are incident to the same vertex $u$, and two of them must overlap each other, which means that $P$ cannot be spherical.

D.2 Proof for Lemma 4

The only-if part: Regard $f$ as the outer face $f^0$ of $G$. Assume that the SPQR-tree of $G$ has an R-node $\nu$ whose skeleton $\sigma(\nu)$ contains a face $f'$ with $V(f') \subseteq V(f)$. Note that $|V(f')| \geq 3$.

Recall that $\sigma(\nu)$ is a triconnected planar graph. Hence $\sigma(\nu)$ contains a vertex $v \in V - V(f')$, since otherwise all vertices are outer vertices of $G$ and appear along the same face of $\sigma(\nu)$ (i.e., $\sigma(\nu)$ is an outerplanar graph), contradicting the triconnectedness of $G$. Choose three vertices $u_1, u_2, u_3$ which are outer vertices of $G$. From the inner vertex $v$, triconnected graph $\sigma(\nu)$ has three internally vertex-disjoint paths $Q_i, i = 1, 2, 3$, each joins $v$ and $u_i$. This means that $G$ also has three internally vertex-disjoint paths $P_i, i = 1, 2, 3$, each joins $v$ and $u_i$, where $P_i$ can be obtained from $Q_i$ by repeatedly replacing virtual edges with real edges.

The if part: Let $f$ be triconnected, and regard $f$ as the outer face $f^0$ of $G$. Then $G$ has three internally vertex-disjoint paths $P_i, i = 1, 2, 3$, each joins an inner vertex $v$ and an outer vertex $u_i$, where we can assume without loss of generality that each $P_i$ contains any other outer vertex than $u_i$. In the subgraph consisting of paths $P_1, P_2, P_3$ and cycle $f^0$, any two vertices in $\{v, u_1, u_2, u_3\}$ are connected by three internally vertex-disjoint paths in $G$. Hence all vertices in $\{v, u_1, u_2, u_3\}$ are contained in the same triconnected component of $G$, which corresponds to an R-node $\nu$ in the SPQR-tree and hence $\{v, u_1, u_2, u_3\}$ appear in the skeleton $\sigma(\nu)$.

D.3 Proof of Lemma 2

Assume that $G(P)$ is not triconnected (since otherwise we are done). To derive a contradiction, we assume that a facial cycle $f_0$ of $G(P)$ is not triconnected. By Lemma 4, the SPQR tree of $G(P)$ has no R-node whose skeleton contains a face $f'_0$ with $V(f'_0) \subseteq V(f_0)$. Regard $G$ as a plane graph with the outer face $f^0 = f_0$.

First consider the case where $G(P)$ has no outer P-node. Then it must have an outer S-node, since $G(P)$ has no other R- and S-nodes. However, in this case, each edge $(u, v)$ in the skeleton $\sigma(\nu)$ is a real edge, since if it corresponds to an R-node then it contains an outer vertex than $u$ and $v$, contradicting the assumption on $G(P)$. Hence $G(P)$ is a simple cycle, and contradicts that $G(P)$ has no vertex of degree 2.

We next consider the case where $G(P)$ has an outer P-node $\nu$ (see Fig. 6 for such an example). Let $u_\nu$ and $v_\nu$ be the vertices in the skeleton of $\nu$. We show by induction that all outer edges must be placed on the same straight line in $P$, which is a contradiction to the nonsingularity. Consider the first and last edges in the skeleton of $\nu$, where each of them corresponds to an outer real edge or an outer S-node, because if it corresponds to an R-node then it contains an outer vertex than $u_\nu$ and $v_\nu$, contradicting to the assumption on $G(P)$. The first inner face $f_1$ (resp., the last inner face $f_2$) in the skeleton $\sigma(\nu)$ of $\nu$ share an edge or at least three vertices with the outer face $f^0$. Let $W_i (i = 1, 2)$ denote the set of the edges or vertices shared by $f^0$ and $f_i$. Then for each $i = 1, 2, f_i$ and $f^0$ are linearly-joined, and the elements in $W_1$ must be placed on the same straight-line $L_i$. Since the points for $u_\nu$ and $v_\nu$ appear in both $L_1$ and $L_2$, it must hold $L_1 = L_2$ and all the elements in $W_1 \cup W_2$ are on the same straight-line $L_1 = L_2$. For an outer S-node child $\mu$ of $\nu$, each edge $e'$ of the skeleton of $\mu$ is an outer real edge or corresponds to an outer P-node $\mu'$. If $\mu'$ corresponds to an outer real edge, then the outer edge must be on the straight-line $L_1$. Consider the case where $\mu'$ corresponds to an outer P-node $\mu'$. In this case, the outer edge $e''$ of the skeleton of $\mu'$ is an outer real edge or corresponds to an outer S-node. In the former case, $e''$ is also placed on $L_1$. By applying this argument repeatedly, we see that all the outer edges of $G(P)$ must be placed on $L_1$, which is a contradiction.

D.4 Proof of Lemma 6

By definition, the image $I_P$ of $P$ on the supporting plane of the bottom face is a straight-line plane drawing of $G(P)$. Note that, for each inner face $f$ of $G$, the region $f$ in $I_P$ is also a star-shaped polygon. To derive a
contradiction, we assume $\xi(G(P)) \geq 2$.

Case 1: There is a pair of outer vertices $u$ and $v$ with $\xi(u, v) \geq 1$. Let $f_1$ and $f_2$ be linearly-joined inner faces at $\{u, v\}$.

First consider the case where $f_1$ and $f_2$ share at least one more vertex, say $w$, than $u$ and $v$. In $P$, $f_1$ and $f_2$ are not coplanar, and thereby, $u, v, w$ and $w$ are contained in the same straight-line in $P$, (Fig. 5(a) shows the case where $u, w$ and $v$ appear in the straight-line in this order). This, however, implies that $w$ is contained in the region $f^*(P)$, and hence is a singular point in $P$, a contradiction to the assumption on $P$.

Second we consider the case where $f_1$ and $f_2$ share an edge $(u, v)$ but no other vertices other than $u$ and $v$. Then all internal points on the line segment for the edge will be contained in the region $f^*(P)$ and thereby they are singular, a contradiction. See Fig. 5(b).

Case 2: There is a pair of vertices $u$ and $v$ with $\xi(u, v) \geq 2$, where one of $u$ and $v$ is an inner vertex of $G$. Let $(f_1, f_2)$ and $(f_3, f_4)$ be two linearly-joined inner faces at $\{u, v\}$. For each $i \in \{1, 2\}$, let $W_i$ be the vertices shared by $f_i$ and $f_{i+1}$. Note that all vertices $W_1 \cup W_3$ must be collinear in $P$ since $f_1$ and $f_2$ (resp., $f_3$ and $f_4$) are not coplanar.

First consider the case where $|W_1|, |W_3| \geq 3$. If all vertices in $W_1 \cup W_3 - \{u, v\}$ are placed on the line segment $[u, v]$ in $P$, then $f_2$ or $f_3$ cannot be a star-shaped polygon (see Fig. 5(c) and (d)). On the other hand, if some vertex $v \in W_1$ (resp., $w \in W_3$) is placed outside the line segment $[u, v]$ in $P$, then $f_1$ (resp., $f_4$) cannot be a star-shaped polygon (see Fig. 6(a)).

Next we consider the case where $|W_i| = 2$ for $i = 1, 2$, without loss of generality we assume that $|W_3| = 2$. Then $f_3$ and $f_4$ share only an edge $(u, v)$ and $|W_1| \geq 3$ (see Fig. 6(b)). In this case, all vertices in $W_1 - \{u, v\}$ must be placed outside the line segment $[u, v]$ in $P$, and face $f_1$ cannot be a star-shaped polygon.

\[\text{D.5 Proofs of Lemmas 7 and 8}\]

We prove Lemmas 7 and 8 by showing that a desired projective transformation can be obtained by combining shear transformation, and "squeeze and lower transformations," two special projective transformations. A projective transformation in $\mathbb{R}^3$ can be described as a linear transformation in homogeneous coordinates of four dimension. A point $p = (x, y, z) \in \mathbb{R}^3$ (resp., an infinite point $p$ of a line parallel to the vector $(x, y, z)$) is denoted by $[p] = [kx, ky, kz, k]$ in homogeneous coordinates (resp., $[p] = [kx, ky, kz, 0]$) for an arbitrary real $k \neq 0$. Then each nonsingular $4 \times 4$ matrix $A$ determines a projective transformation that maps a point $[p]$ to the point $[p'] = A \cdot [p]$. Equivalently, given two sets of arbitrary five points $S = \{p_1, p_2, p_3, p_4, p_5\}$, $S' = \{p'_1, p'_2, p'_3, p'_4, p'_5\} \subset \mathbb{R}^3$, where no four points in $S$ or $S'$ are on the same plane, a projective transformation that maps each $p_i \in S$ to $p'_i \in S'$ is uniquely determined. In general, a projective transformation may not preserve the convexity of polyhedra or the structure of the vertex-edge graph. We say that a projective transformation $\psi$ has the convex property for a set $S \subset \mathbb{R}^3$ of points if, for any line segment $[q_1, q_2] \subset S$, the point $q = \lambda q_1 + (1 - \lambda) q_2$ (for $0 \leq \lambda \leq 1$) of a convex combination of $q_1$ and $q_2$ is mapped to a point $\psi(q) = \delta \psi(q_1) + (1 - \delta) \psi(q_2) \in S$ (for $0 \leq \delta \leq 1$), i.e., a convex combination of $\psi(q_1)$ and $\psi(q_2)$. In this case, $\psi$ transforms a convex polyhedron $S \subset S$ into a convex polyhedron $P \subset S$ with $G(P') = G(P)$.

**Proof of Lemma 7**

Let $P_0$ be a convex polyhedron with $G(P_0) = G$. Such a convex polyhedron $P_0$ exists by Steinitz’ theorem. Let $f_b, f_t \in F$ be two adjacent faces of $G$ designated as the base and top faces. Let $Y = \Delta(a, b, c, d)$ be a given type A pyramid. In what follows, we assume that $H(a, b, c, d)$ is contained in the $xy$-plane, and we will make all faces $f \in F - \{f_b\}$ of a desired polyhedron visible from any specified view direction over the $xy$-plane.

We give a sequence of projective transformations (a)-(d), which transforms $P_0$ into a convex polyhedron $P$ with $G(P) = G$ that fits into $Y$. Let $\theta$ denote the interior angle made by the base and top faces in $Y$. The interior angle made by faces $f_b$ and $f_t$ is denoted by $\angle f_b, f_t$.

(a) Set the length of the base edge to be equal to that of edge $(a, b)$ in $Y$ (we scale up or down $P_0$ if necessary). We then place $P_0$ on the $xy$-plane and the base edge on the $x$-axis, as in Fig. 17(a). Now the current $\angle f_b, f_t$ may be less than $\pi/2$ or larger than $\pi/2$. We apply shear transformation (normal to $y$-axis) to $P_0$ so that the resulting convex polyhedron $P_1$ satisfies $\angle f_b, f_t = \pi/2$, as shown in Fig. 17(b). Shear transformation with a real $\alpha \in \mathbb{R}$ normal to $y$-axis maps a point $q = (x, y, z) \in \mathbb{R}^3$ to point $q' = (x, y + \alpha z, z)$. Shear transformation is an affine transformation (a linear transformation on $\mathbb{R}^3$), and has the convex property for $\mathbb{R}^3$. Hence the resulting polyhedron $P_1$ is convex and satisfies $G(P_1) = G(P_0)$.

(b) We place $P_1$ on the $xy$-plane and the base edge on the $x$-axis, as in Fig. 18(a). Then we transform $P_1$ so that the polygon $f_t$ is contained in the triangle $(a, b, d)$ of $Y$. For this, we “squeeze” part of the polygon $f_t$ while keeping the base edge $e = (u, v)$ unchanged.
Figure 17: A shear transformation normal to the y-axis.

Squeeze transformation with reals $0 < \alpha, \beta < 1$ normal to y-axis maps a point $q = (x, y, z) \in \mathbb{R}^3$ to point

$$q' = (x', y', z') = \left( \frac{x}{1 + y/\beta}, \frac{\alpha(1 + 1/\beta) y}{1 + y/\beta}, \frac{z}{1 + y/\beta} \right).$$

(1)

Squeeze transformation is a projective transformation which uses transformation matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha(1 + 1/\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/\beta & 0 & 1 \end{pmatrix}.$$ 

(2)

We show that squeeze transformation normal to y-axis has the convex property for the set $\{q = (x, y, z) \in \mathbb{R}^3 \mid y \geq 0 \}$. For two points $q_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, 2$, a point $q = \lambda q_1 + (1 - \lambda) q_2$ ($0 \leq \lambda \leq 1$) is mapped to point

$$q' = \left( \frac{\lambda x_1 + (1 - \lambda) x_2}{1 + (\lambda y_1 + (1 - \lambda) y_2)/\beta}, \frac{\alpha(1 + 1/\beta)(\lambda y_1 + (1 - \lambda) y_2)}{1 + (\lambda y_1 + (1 - \lambda) y_2)/\beta}, \frac{\lambda z_1 + (1 - \lambda) z_2}{1 + (\lambda y_1 + (1 - \lambda) y_2)/\beta} \right) = \delta q_1 + (1 - \delta) q_2,$$

where

$$\delta = \frac{\lambda(1 + y_1)/\beta}{1 + (\lambda y_1 + (1 - \lambda) y_2)/\beta},$$

which satisfies $0 \leq \delta \leq 1$ if $y_1, y_2 \geq 0$, as required. We easily see from (1) that a squeeze transformation with sufficiently small $\alpha, \beta > 0$ transforms $P_1$ into a convex polyhedron $P_2$ with $G(P_2) = G(P_1)$ in which the polygon $f_t$ is contained in the triangle $(a, b, d)$ of $Y$, as illustrated in Fig. 18(c). Note that the polygon $f_t$ remains unchanged in $P_2$.

Figure 18: A squeeze transformation normal to the y-axis.
(c) We place $P_2$ on the $xy$-plane so that an internal point (such as the barycenter) of the polygon $f_b$ will be on the origin of the $xyz$-coordinates, as in Fig. 19(a). Then we transform $P_2$ into a convex polyhedron $P_3$ with $G(P_3) = G(P_2)$ by a squeeze transformation normal to $z$-axis. Squeeze transformation normal to $z$-axis has the convex property for the set $\{q = (x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. For sufficiently small $\alpha, \beta > 0$, all faces except for $f_b$ will be visible in $P_3$ and $\angle f_b, f_i$ will be smaller than $\theta$, as in Fig. 19(c). After this, we use shear transformation normal to $y$-axis to make $\angle f_b, f_i$ exactly $\theta$ (note that all faces except for $f_b$ remain visible in this sheared polyhedron $P_3$, since the polygon $f_b$ has been squeezed and $\angle f_b, f_i < \theta$ hold before applying the shear transformation). Note that $f_b$ remains unchanged in $P_3$.

Figure 19: A squeeze transformation normal to the $z$-axis.

(d) The current polyhedron $P_3$ satisfies $G(P_3) = G$ and $\angle f_b, f_i = \theta$, and the polygon $f_b$ is contained in the triangle $(a, b, d)$ of $Y$. Finally, we lower the height of $P_3$ while the polygon $f_b$ and $\angle f_b, f_i = \theta$ remain unchanged. We can lower the height of polyhedron $P_3$ as much as possible by using “lower” transformation. We place $P_3$ on the $xy$-plane and the base edge on the $x$-axis, as in Fig. 20(a).

Lower transformation is a projective transformation that maps a point $q = (x, y, z)$ to point $q' = (x, y - \alpha z, \alpha z / \tan \theta)$, which preserves the polygon $f_b$ and the angle $\angle f_b, f_i = \theta$. Lower transformation $\psi$ maps a point $q = \lambda q_1 + (1 - \lambda)q_2$ ($0 \leq \lambda \leq 1$) to a point $\psi(q) = \lambda \psi(q_1) + (1 - \lambda)\psi(q_2)$, and has the convex property for $\mathbb{R}^3$. Moreover, this shows that each face will be flattened as much as possible as the height decreases. Let $P_4$ be the resulting convex polyhedron $P_3$ in which $G(P_4) = G$, $\angle f_b, f_i = \theta$, the length of the base edge of $P_4$ is equal to that of $(a, b)$, and the polygon $f_b$ is contained in the triangle $(a, b, c)$ of $Y$. To construct $P_4$, we lower the height of $P_3$ so that (i) the polygon $f_b$ is contained in the triangle $(a, b, d)$ of $Y$, and (ii) all faces $f \in F - \{f_b\}$ will be visible from a given view direction from above, where the view direction is determined by the original $Y$. Therefore, $P_4$ is a desired polyhedron in Lemma 7.

Figure 20: A projective transformation for lowering the height of a convex polyhedron.

**Proof of Lemma 8**

Let $G = (V, E, F)$ be a triconnected planar graph with two adjacent faces $f_{a1}$ and $f_{a2}$ designated as the two side faces. Let $Y = \Delta(a, b, c, d)$ be a type B pyramid in $\mathbb{R}^3$, and $\theta$ denote the interior angle made by the two
side faces in $Y$. The interior angle made by faces $f_1$ and $f_2$ is denoted by $\angle f_1, f_2$. As in the case of type A pyramid, we again assume that $H(a, b, d) = G$ and we will construct a polyhedron in which all faces $f \in F - \{f_i\}$ are visible to any specified view direction from which the two side faces of $Y$ are invisible. Such a polyhedron can be constructed in the same way as the above transformations (a)-(d) by regarding $f_1$ and $f_2$ as $f_i$ and $f_j$, respectively. Let $P_3$ be the polyhedron after (c), where $G(P_3) = \theta$ and $\angle f_1, f_2 = \theta$, and the polygon $f_1$ is contained in the triangle $(a, b, d)$ of $Y$.

To obtain a desired polyhedron $P_4$ from $P_3$ in (d), we lower the height of $P_3$ so that (i) the polygon of $f_2$ is contained in the triangle $(a, b, d)$ of $Y$, and (ii) all faces $f \in F - \{f_1, f_2\}$ will be visible from a given view direction $\tau$ from which the two side faces of $Y$ are invisible. Since the interior side of each side face of $Y$ is visible from the view direction $\tau$, each face $f \in F - \{f_1, f_2\}$ will be visible from $\tau$ before it becomes flat on the $xy$-plane as the height of $P_3$ decreases. Therefore, we can obtain a desired polyhedron $P_4$ in Lemma 8.

E Some examples of polyhedra and biconnected plane graphs

Figure 21: Examples of polyhedra: (a) a nonspherical polyhedra $P_a$ with $\kappa(G(P_a)) = 0$ obtained from a cube $(v_1, \ldots, v_4; v_5, \ldots, v_8)$ by placing a pyramid $\Delta(v_9, v_{10}, v_{11}, v_{12})$ on the face $(v_2, v_3, v_7, v_6)$ without touching any vertex or edge of the face, where the face containing edges $(v_2, v_3)$ and $(v_{10}, v_{11})$ of $P_a$ is not a simple polygon; (b) a nonspherical polyhedra $P_b$ with $\kappa(G(P_b)) = 1$ obtained from a cube $(v_1, \ldots, v_4; v_5, \ldots, v_8)$ by placing a pyramid $\Delta(v_6, v_9, v_{10}, v_{11})$ on the face $(v_2, v_3, v_7, v_6)$ sharing only vertex $v_6$ with the face, where the face containing the edge $(v_2, v_3)$ is not a simple polygon; (c) a spherical polyhedra $P_c$ with $\kappa(G(P_c)) = 2$ obtained from $P_b$ by placing another pyramid $\Delta(v_7, v_{10}, v_{12}, v_{13})$ on the face $(v_6, v_2, v_3, v_7, v_6, v_{10}, v_9)$, where $P_c$ is not strictly bended because faces $(v_2, v_3, v_7, v_{12}, v_{10}, v_9, v_6)$ and $(v_6, v_{10}, v_7)$ are coplanar.
Figure 22: Examples of upward polyhedra: (1) a proper upward polyhedron $P_d$ obtained from a cube $(v_1, \ldots, v_4, v_5, \ldots, v_8)$ by placing a pyramid $\Delta(v_9, v_{10}, v_{11}, v_{12})$ on the face $(v_2, v_3, v_7, v_6)$, where for each concave edge, there is a non-convex face that contains both endvertices at the same time. (2) a proper upward polyhedron $P_e$ obtained from $P_d$ by placing a pyramid $\Delta(v_{10}, v_{12}, v_{13}, v_{14})$ on face $(v_2, v_3, v_7, v_6, v_{10}, v_{12}, v_9)$, where no non-convex face contains the concave edge $(v_{10}, v_{12})$, but the non-convex face $(v_2, v_3, v_7, v_6, v_{10}, v_{12}, v_9)$ contains both endvertices $v_{10}$ and $v_{12}$. (3) an upward polyhedron $P_f$ obtained from a cube $(v_1, \ldots, v_4, v_5, \ldots, v_8)$ by removing a pyramid $\Delta(v_9, v_{10}, v_{11}, v_{12})$, where $P_f$ is not proper although $G(P_f) = G(P_d)$, because no face in $P_f$ contains both endvertices of concave edge $(v_{11}, v_{12})$.

Figure 23: (a) A polyhedron $P'_1$ obtained from a cube $(v_1, \ldots, v_4, v_5, \ldots, v_8)$ by removing a pyramid $\Delta(v_2, v_6, v_9, v_{10})$; (b) a polyhedron $P'_2$ obtained from $P'_1$ by attaching a pyramid on the face $(v_2, v_3, v_7, v_6, v_{10})$ and sharing the convex edge $(v_2, v_{10})$; (c) a polyhedron $P'$ whose vertex-edge graph is given by the graph $G'$ in Fig. 25(a), where $P'$ is obtained from $P'_2$ by placing a pyramid on the face $(v_2, v_3, v_7, v_6, v_{10}, v_{11}, v_{12})$ sharing part of the concave edge $(v_{10}, v_{11})$.

Figure 24: A spherical polyhedron $P''$ in (a) and (c), where the vertex-edge graph is given by the graph $G''$ in Fig. 25(b), and the polyhedron is obtained from a cube $(v_1, \ldots, v_4, v_5, \ldots, v_8)$ by placing four pyramids as shown in (b).
Figure 25: Examples of non upward-polyhedral graphs: (a) a bi-connected plane graph $G'$ with $\xi(G') = 2$ which has a spherical polyhedron $P'$ with $G(P') = G'$ and a non-star-shaped face $(v_2, v_3, v_7, v_6, v_10)$ in Fig. 23(c); (b) a bi-connected plane graph $G''$ with $\xi(G'') = 3$ which has a spherical polyhedron $P''$ with $G(P'') = G''$ and non-visible faces in Fig. 24(a) and (c); (c) a bi-connected plane graph $G'''$ with $\xi(G''') = 3$ which has no spherical polyhedron $P$ with $G(P) = G'''$ since the separation pair $\{v_7, v_6\}$ with $\xi(v_6, v_7) = 3$ satisfies the condition of Lemma 3.