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Minimum Augmentation of a Multigraph to an ℓ-Edge and 3-Vertex Connectivity Multigraph [1]

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AbstractGiven an undirected multigraph $G = (V, E)$ and two positive integers ℓ and k, the edge-and-vertex connectivity augmentation problem asks to augment G by the smallest number of new edges so that the resulting multigraph becomes ℓ-edge-connected and k-vertex-connected. In this paper, we show that the problem can be solved in polynomial time for any fixed ℓ and $k = 3$.

Keywords: undirected graph, multigraph, edge-connectivity, vertex-connectivity, graph augmentation, network design, edge splitting, minimum cut, polynomial time deterministic algorithm.

Abbreviated title: Minimum Augmentation to ℓ-Edge and 3-Vertex Connectivity

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1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*, where we denote $|V|$ by n (or by $n(G)$) and the number of pairs of vertices which are adjacent in G by m (or by $m(G)$). An edge with end vertices u and v is denoted by (u, v). Throughout the paper, an undirected multigraph is called a graph unless a confusion arises. For a nonnegative integer ℓ, a multigraph G is called *ℓ-edge-connected* if the deletion of any $\ell - 1$ or fewer edges leaves a connected multigraph. For a nonnegative integer k, a multigraph G is called *k-vertex-connected* if $|V| \geq k + 1$ and the deletion of any $k - 1$ or fewer vertices leaves a connected multigraph.

The problem of augmenting a graph by adding the smallest number of new edges to meet edge-connectivity or vertex-connectivity requirements has been extensively studied as an important subject in the network design problem, the data security problem [17], the graph drawing problem [16] and others, and many efficient algorithms have been developed so far.

In a communication network where both link and node failures can occur, it seems to natural to consider the network design problem that handles both the edge- and vertex-connectivities. Hsu and Kao [10] first treated the problem of augmenting the edge- and vertex-connectivities simultaneously, and presented a liner time algorithm to the problem of augmenting a multigraph $G = (V, E)$ with two specified vertex sets $X, Y \subseteq V$ by adding a minimum number of edges such that two vertices in X (resp., in Y) are 2-edge-connected (resp., 2-vertex-connected). Given a multigraph $G = (V, E)$ and two positive integers ℓ and k, Ishii *et al.* [14] consider the problem of augmenting G by the smallest number of new edges to obtain an ℓ-edge-connected and k-vertex-connected multigraph (we call such a multigraph (ℓ, k)-connected). This problem is called the *edge- and vertex-connectivity augmentation problem*, and is denoted by EVAP(ℓ, k) (where $\ell \geq k$ is assumed since any k-vertex-connected graph is k-edge-connected).

EVAP$(\ell, 1)$ is called the *edge-connectivity augmentation problem.* Watanabe and Nakamura [22] first showed that the edge-connectivity augmentation problem can be solved in polynomial time for an arbitrary ℓ , and the complexity for solving EVAP$(\ell, 1)$ has been improved [6, 7, 20]. EVAP(k, k) is called the *vertex-connectivity augmentation problem.* It is still open whether the vertex-connectivity augmentation problem is polynomially solvable for an arbitrary k, even if an initial graph G is $(k - 1)$-vertex-connected. For small k, polynomial time algorithms are known (see [3, 12] for $k = 2$, [11, 23, 24] for $k = 3$ and [9] for $k = 4$). Ishii *et al.* [14] proved that EVAP$(\ell, 2)$ is polynomially solvable. For an arbitrary ℓ and $k = 3$, they showed [15] that EVAP$(\ell, 3)$ can be solved in polynomial time, under the assumption that ℓ is a fixed constant and the initial multigraph is already 2-vertex-connected. However, it is left open whether EVAP$(\ell, 3)$ with a fixed ℓ can be solved in polynomial time or not when a given multigraph G is not 2-vertex-connected.

In this paper, we consider EVAP$(\ell, 3)$ for an arbitrary initial multigraph $G = (V, E)$, which is not necessarily 2-vertex-connected, and a positive integer ℓ. An algorithm, called EV-AUGMENT3, which optimally augments a 2-vertex-connected graph to an $(\ell, 3)$-connected graph is obtained [15]. However, it seems difficult to extend directly the algorithm to the case

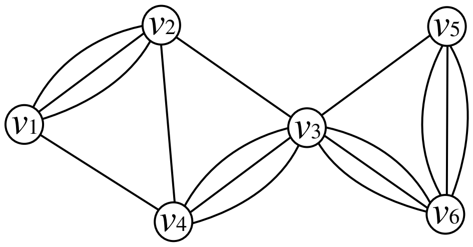

Figure 1: EVAP(5,3) for a graph G, which can be made $(5,3)$-connected optimally by adding two new edges (v_1, v_6) and (v_2, v_5). If we first solve EVAP(5,2) for G, then we may obtain a $(5,2)$-connected multigraph G' by adding a set $\{(v_2, v_5), (v_2, v_5)\}$ of two edges to G, and we need one more edge to make G' $(5,3)$-connected.

where an initial graph G may not be 2-vertex-connected, since many properties used in EV-AUGMENT3 heavily depend on the 2-vertex-connectivity of G. Alternatively, one may first solve EVAP$(\ell, 2)$ for G as an intermediate step (where an algorithm, called EV-AUGMENT, which optimally augments an arbitrary multigraph to an $(\ell, 2)$-connected graph is obtained [14]) and then solve EVAP$(\ell, 3)$ for the resulting $(\ell, 2)$-connected graph G' to obtain an $(\ell, 3)$-connected graph G''. However, in this case, the resulting multigraph G'' may not be optimally augmented from the original multigraph G, as examplified in Figure 1. For the graph G in the figure, a set of two multiple edges (v_2, v_5) is an optimal solution to EVAP(5,2) for G, but a unique optimal solution to EVAP(5,3) is given by a set of two edges (v_1, v_6) and (v_2, v_5) (hence we cannot obtain an optimal solution to EVAP(5,3) once G is augmented with two edges of (v_2, v_5)).

In this paper, we first derive two lower bounds $\lceil \alpha_{\ell,k}(G)/2 \rceil$ and $\beta_{\ell,k}(G)$ on the optimal value of EVAP(ℓ, k), denoted by $opt_{\ell,k}(G)$. We next obtain an $(\ell, 2)$-connected multigraph $G_2 = (V, E \cup F)$ such that $|F| = \lceil \alpha_{\ell,3}(G)/2 \rceil$. We show that such G_2 can be computed by applying EV-AUGMENT [14]. We then apply several procedures used in EV-AUGMENT3 [15] in order to replace some edges in F with the same number of edges to attain the 3-vertex-connectivity while preserving its $(\ell, 2)$-connectivity. However, those procedures cannot be directly applied to a graph unless it is 2-vertex-connected. To remedy this, we show that there always exist some edges in F for which the procedures are applicable. As a result of applying these procedures, we can show that either $opt_{\ell,3}(G) = \max\{\lceil \alpha_{\ell,3}(G)/2 \rceil, \beta_{\ell,3}(G)\}$ or $opt_{\ell,3}(G) \leq 5\ell/2 - 1$ holds and that a set of the smallest number of new edges can then be constructed in $O((5\ell/2 - 1)n^{5\ell+1} + n^2 m + n^3 \log n)$ time. Furthermore, we can show that if $\delta_{\ell,3}(G) \geq \ell$ or $opt_{\ell,3}(G) \geq 5\ell/2$ holds, then an optimal solution can be found in $O(n^2 m + n^3 \log n)$ time, and if $opt_{\ell,3}(G) \leq 5\ell/2 - 1$ holds, then a feasible solution F' such that $|F'| \leq \min\{2opt_{\ell,3}(G) - 1, 5\ell/2 - 1, opt_{\ell,3}(G) + (\ell + 1)/2\}$, can be found in $O(n^2 m + n^3 \log n)$ time. The entire algorithm is called EV-AUGMENT3*.

In Section 2, after introducing basic definitions, we derive a lower bound on the optimal value of EVAP$(\ell, 3)$ for an arbitrary multigraph G. In Section 3, we review the concept of edge-splitting and give an outline of EV-AUGMENT3*. In Sections 4 – 8, we prove the correctness of each step in our algorithm. In Section 9, we state a concluding remark.

2 Main Theorem

In a graph $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". A subset X *intersects* another subset Y if none of subsets $X \cap Y$, $X - Y$ and $Y - X$ is empty. A family $\mathcal{X}$ of subsets $X_1, \ldots, X_p$ is called *laminar* if no two subsets in $\mathcal{X}$ intersect each other (possibly $X_i \subseteq X_j$ for some $X_i, X_j \in \mathcal{X}$). We say that a subset X *crosses* another subset Y if they intersect each other and in addition $V - (X \cup Y) \neq \emptyset$ holds. A *partition* $X_1, \ldots, X_t$ of the vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset V' of V.

For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in G, $G[V']$ denotes the subgraph induced by V'. For $V' \subset V$ (resp., $E' \subseteq E$), we denote subgraph $G[V - V']$ (resp., $(V, E - E')$) by $G - V'$ (resp., $G - E'$). For $E' \subset E$, we denote $V(G[E'])$ by $V[E']$. For an edge set E' with $E' \cap E = \emptyset$, we denote the augmented graph $G = (V, E \cup E')$ by $G + E'$. For a vertex set V' in G, we define $E_G[V'] = \{e \in E(G) | V[e] \cap V' \neq \emptyset\}$. For two disjoint subsets of vertices $X,\ Y \subset V$, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $c_G(X, Y)$. In particular, $E_G(u, v)$ is the set of edges with end vertices u and v.

A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is defined by $c_G(X, V - X)$, which may also be written as $c_G(X)$. In particular, $c_G(v)$ for a vertex v is the *degree* of v. Let $\delta(G)$ denote the minimum degree of G. A cut with the minimum size is called a (*global*) *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. The *local edge-connectivity* $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in G that separates x and y (i.e., $|\{x, y\} \cap X| = 1$), or equivalently the maximum number of edge-disjoint paths between x and y by Menger's theorem.

For a subset X of V, a vertex $v \in V - X$ is called a *neighbor* of X if it is adjacent to some vertex $u \in X$, and the set of all neighbors of X is denoted by $\Gamma_G(X)$. A maximal connected subgraph G' in a graph G is called a *component* of G (for notational convenience, a component H may be represented by its vertex set $X = V(H)$), and denote the number of components in G by $p(G)$.

A *disconnecting set* of G is defined as a subset S of V such that $p(G - S) > p(G)$ holds and no $S' \subset S$ has this property. The *local vertex-connectivity* $\kappa_G(x, y)$ for two vertices $x, y \in V$ is defined to be the number of internally-disjoint paths between x and y in G. Also by Menger's theorem, $\kappa_G(x, y)$ for nonadjacent vertices x and y is equal to the minimum size of a disconnecting set S that disconnects x and y (i.e., x and y is contained in distinct components in $G - S$). Let $\hat{G}$ denote the simple graph obtained from G by replacing multiple edges in $E_G(u, v)$ by a single edge (u, v) for all $u, v \in V$. A component G' of G with $|V(G')| \geq 3$ always has a disconnecting set unless $\hat{G}'$ is a complete graph. For a connected G, a disconnecting set of the minimum size is called a *minimum disconnecting set*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G; we define $\kappa(G) = 0$ if G is not connected, and $\kappa(G) = n - 1$ if G is a complete graph. A subset $T \subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum disconnecting set in G. A tight set D is called *minimal* if no proper subset D' of D is tight (hence a minimal tight set D induces a connected subgraph $G[D]$). We denote a family of all minimal tight sets in G by $\mathcal{D}(G)$, and the maximum number of pairwise disjoint minimal tight sets in G by $t(G)$. Hence $|\mathcal{D}(G)| \geq t(G)$.

The following equalities hold for two crossing cuts X and Y in $G = (V, E)$:

$$\begin{aligned} c_G(X) + c_G(Y) &= c_G(X \cap Y) + c_G(X \cup Y) + 2c_G(X - Y, Y - X), \\ c_G(X) + c_G(Y) &= c_G(X - Y) + c_G(Y - X) + 2c_G(X \cap Y, V - (X \cup Y)). \end{aligned} \tag{2.1}$$

The following equality holds for two cuts X and Y in $G = (V, E)$:

$$|\Gamma_G(X)| + |\Gamma_G(Y)| \geq |\Gamma_G(X \cap Y)| + |\Gamma_G(X \cup Y)|. \tag{2.2}$$

Observe that a graph G with $\kappa(G) = 2$ has the following property.

Lemma 2.1 *If $\kappa(G) = 2$, then any two minimal tight sets X and Y in G are pairwise disjoint (i.e., $|\mathcal{D}(G)| = t(G)$).* ☐

For a nonnegative integer ℓ (resp., k), $G = (V, E)$ is called *ℓ-edge-connected* (resp., *k-vertex-connected*) if $\lambda_G(x, y) \geq \ell$ (resp., $\kappa_G(x, y) \geq k$) holds for every $x, y \in V$.

Several algorithms have been developed for EVAP(ℓ, k). These algorithms use the following lower bound on the optimal value $opt_{\ell,k}(G)$ of the EVAP(ℓ, k). We define

$$\alpha_{\ell,k}(G) = \max_{\text{all subpartitions } \mathcal{X}} \left\{ \sum_{i=1}^{q_1} (\ell - c_G(X_i)) + \sum_{i=q_1+1}^{q_2} (k - |\Gamma_G(X_i)|) \right\}, \tag{2.3}$$

where the maximum is taken over all subpartitions $\mathcal{X} = \{X_1, \ldots, X_{q_1}, X_{q_1+1}, \ldots, X_{q_2}\}$ of V with $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = q_1 + 1, \ldots, q_2$, and

$$\beta_{\ell,k}(G) = \max_{\substack{\text{all vertex subsets} \\ S \text{ with } |S| = k-1}} \left\{ p(G - S) - 1 + \max\left\{ \max_{v \in S}\{\ell - c_G(v)\}, 0 \right\} \right\}. \tag{2.4}$$

Let

$$\gamma_{\ell,k}(G) = \max\{\lceil \alpha_{\ell,k}(G)/2 \rceil, \beta_{\ell,k}(G)\}.$$

It is observed in [15] that $\gamma_{\ell,k}(G)$ is a lower bound on $opt_{\ell,k}(G)$. Ishii *et al.* [14] showed that EVAP$(\ell, 2)$ can be solved in $O((nm + n^2 \log n) \log n)$ time by adding $\gamma_{\ell,2}(G)$ new edges for an arbitrary ℓ. But $\gamma_{\ell,k}(G)$ can be smaller than the optimal value for an arbitrary $\ell > k \geq 3$. Ishii *et al.* [15] proved that if an initial graph G is 2-vertex-connected, then EVAP(ℓ,3) can be solved in $O(n^4)$ time for a fixed ℓ by adding at most $\max\{\gamma_{\ell,3}(G), 5\ell/2 - 1\}$ new edges.

In this paper, we use the lower bound $\gamma_{\ell,3}(G) = \max\{\lceil \alpha_{\ell,3}(G)/2 \rceil, \beta_{\ell,3}(G)\}$ and show the next result.

Theorem 2.1 *Let G be an arbitrary multigraph with n vertices and m adjacent vertex pairs.*
(1) *For any integer $\ell \geq 4$, $\gamma_{\ell,3}(G) \leq opt_{\ell,3}(G) \leq \max\{\gamma_{\ell,3}(G), 5\ell/2-1\}$ holds and an optimal solution to* EVAP$(\ell,3)$ *can be found in $O((5\ell/2-1)n^{5\ell+1}+n^2m+n^3\log n)$ time.*
(2) *If $\delta(G) \geq \ell$ or $\gamma_{\ell,3}(G) \geq 5\ell/2$, then $opt_{\ell,3}(G) = \gamma_{\ell,3}(G)$ holds and an optimal solution F can be found in $O(n^2m+n^3\log n)$ time.*
(3) *If $\gamma_{\ell,3}(G) \leq 5\ell/2-1$, then a feasible solution F' to* EVAP$(\ell,3)$ *such that $|F'| \leq \min\{2opt_{\ell,3}(G) - 1, 5\ell/2-1, opt_{\ell,3}(G)+\ell/2-1\}$, can be found in $O(n^2m+n^3\log n)$ time.* □

3 Algorithm

3.1 Edge-Splitting

Here we review an operation of *edge-splitting*. Given a multigraph G, a designated vertex $s \in V(G)$, vertices $u, v \in \Gamma_G(s)$ (possibly $u = v$) and a nonnegative integer $d \leq \min\{c_G(s,u), c_G(s,v)\}$, we construct graph G' by deleting d edges from $E_G(s,u)$ and $E_G(s,v)$, respectively, and adding new d edges to $E_G(u,v)$: $c_{G'}(s,u) := c_G(s,u) - d$, $c_{G'}(s,v) := c_G(s,v) - d$, $c_{G'}(u,v) := c_G(u,v) + d$, and $c_{G'}(x,y) := c_G(x,y)$ for all other pairs $x, y \in V(G)$. In the case $u = v$, we interpret that $c_{G'}(s,u) := c_G(s,u) - 2d$, $c_{G'}(u,u) := c_G(u,u) + d$, and $c_{G'}(x,y) := c_G(x,y)$ for all other pairs $x, y \in V(G)$, where an integer d is chosen so as to satisfy $0 \leq d \leq \frac{1}{2}c_G(s,u)$. We say that G' is obtained from G by *splitting* d pair of edges (s,u) and (s,v) (or by splitting (s,u) and (s,v) by size d). A sequence of splittings is *complete* if the resulting graph G' does not have any neighbor of s. Also it is called *loopless* if it contains no splitting of two edges (s,u) and (s,u) (i.e., the degree of any vertex in $V(G) - s$ remains unchanged).

The following theorem is proven by Lovász [18].

Theorem 3.1 [6, 18] *Let G be a multigraph with a designated vertex $s \in V(G)$ with even $c_G(s)$, and $\ell \geq 2$ be an integer such that $\lambda_G(x,y) \geq \ell$ for all pairs $x, y \in V(G) - s$. Then for each $u \in \Gamma_G(s)$ there is a vertex $v \in \Gamma_G(s)$ such that the graph G' obtained by splitting one pair of edges (u,s) and (s,v) satisfies $\lambda_{G'}(x,y) \geq \ell$ for all pairs $x, y \in V(G) - s$.* □

Repeating the splitting in this theorem, we see that, if $c_G(s)$ is even, there always exists a complete splitting at s such that the resulting graph G' satisfies $\lambda_{G'-s}(x,y) \geq \ell$ for every pair of $x, y \in V(G) - s$. It is shown that such a complete splitting at s can be computed in $O((m + n\log n)n\log n)$ time [20, 21] and the number of new pairs of vertices which became adjacent after the complete splitting can be bounded by $O(n)$ [1].

3.2 Algorithm for EVAP(ℓ,3)

We present an algorithm, called EV-AUGMENT3*, for solving EVAP$(\ell,3)$ with an arbitrary multigraph and a integer ℓ. A pair $\{v_1, v_2\}$ of vertices is called a *disconnecting pair* in G if $S = \{v_1, v_2\}$ is a disconnecting set in G. For a disconnecting pair S, there is the component X of G (where G may not be connected) such that $X \supseteq S$ by the minimality of S. For such X, we call the components in $G[X] - S$ *the S-components*. If $\kappa(G) = 2$, then any S-component of a disconnecting pair S is also a tight set. For a graph $G=(V,E)$, let $\mathcal{P}_3(G)$ denote the set of all vertex pairs x, y such that $\kappa_G(x,y)\geq 3$. Thus $\mathcal{P}_3(G)=\binom{V}{2}$ if $\kappa(G)\geq 3$. An operation of removing a subset $F'\subseteq E$ and adding a set F'' of new edges with $|F''|=|F'|$ is called a *shifting*, and is denoted by F''/F'. In particular, a shifting F''/F' is called a *switching* if it does not change degree $c_G(v)$ of any vertex v in G. Given an ℓ-edge-connected graph G, a sequence of switchings or shiftings of edges is called *improving* in G if the resulting graph G' remains ℓ-edge-connected and $\mathcal{P}_3(G')\supseteq\mathcal{P}_3(G)$ holds. An edge $e=(u,w)$ is called *switching-admissible* (abbreviated to *s-admissible* in the subsequent discussion) with respect to a disconnecting pair S if $S\cap\{u,w\}=\emptyset$ and $p(G-S)=p((G-S)-e)$ hold. Define $E^*(H)=\{e\in E(H)|\kappa(H-e)<2\}$ for a multigraph H with $\kappa(H)\geq 2$.

The algorithm EV-AUGMENT3* consists of the following five major steps. In each step, we also describe some properties which verify its correctness. The proof for these properties will be given in the subsequent sections. Figure 2 illustrates the process of these five steps for an example multigraph.

Algorithm EV-AUGMENT3*

Input: An undirected multigraph $G=(V,E)$ ($|V|\geq 4$) and an integer $\ell\geq 4$.

Output: A set F of new edges with $|F|=opt_{\ell,3}(G)$ such that $G+F$ is $(\ell,3)$-connected.

Step I (Addition of vertex s and associated edges): Add a new vertex s together with a set F_1 of new edges between s and V such that the resulting $G_1=(V\cup\{s\},E\cup F_1)$ satisfies

$$\begin{aligned} c_{G_1}(X)\geq\ell \qquad & \text{for all cuts } X\subset V, && (3.1)\\ |\Gamma_G(X)|+|\Gamma_{G_1}(s)\cap X|\geq 3 \qquad & \text{for all cuts } X\subset V \text{ with } V-X-\Gamma_G(X)\neq\emptyset && \\ & \text{and } |\Gamma_G(X)|+|X|\geq 3, && (3.2)\\ |\Gamma_G(x)|+c_{G_1}(s,x)\geq 3 \qquad & \text{for all vertices } x\in V, && (3.3)\end{aligned}$$

and $|F_1|$ is *minimum* subject to (3.1) – (3.3). We describe in Section 4 how to find such F_1 in polynomial time.

Property 3.1 $|F_1|=\alpha_{\ell,3}(G)$. □

If $c_{G_1}(s)$ is odd, then we add to F_1 a new edge $\hat{e}=(s,w)$ for an arbitrary vertex $w\in V$ so that $c_{G_1}(s)$ becomes even, where we call such edge $\hat{e}=(s,w)$ *extra*.

Step II (Edge-splitting): For G, we use the following property.

Property 3.2 *There is a complete and loopless splitting at s in G_1 such that the resulting multigraph $G_2=(V,E\cup F_2)$, where s is ignored, is $(\ell,2)$-connected.* □

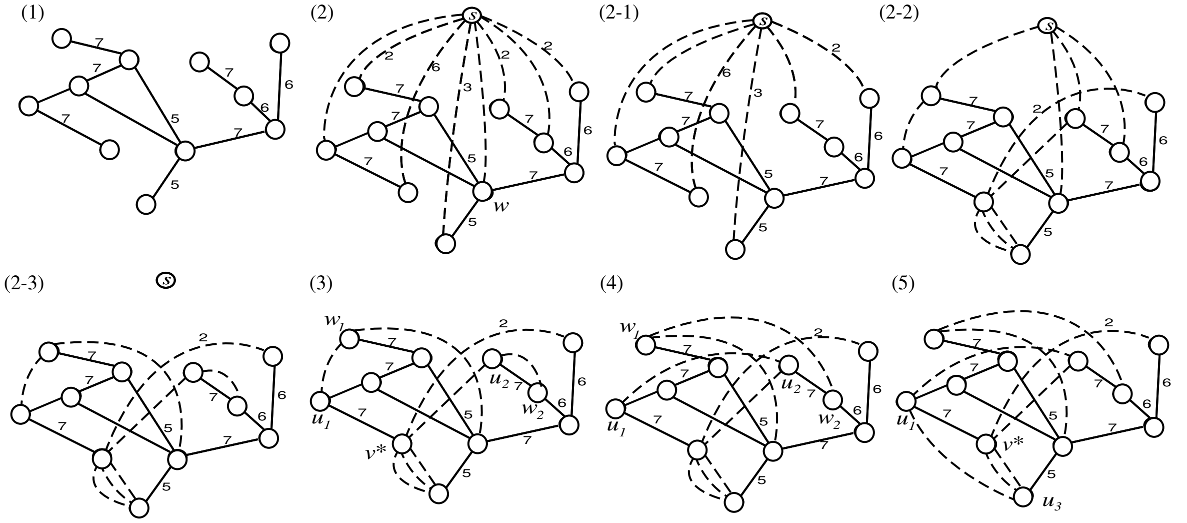

Figure 2: Computational process of algorithm EV-AUGMENT3* for $\ell = 8$. **(1)** An initial multigraph $G = (V, E)$ with $\lambda(G) = \kappa(G) = 1$, where the number beside each edge is the multiplicity of the edge (the numbers for multiplicity 1 are omitted). The two lower bounds in Section 2 are $\lceil \alpha_{8,3}(G)/2 \rceil = \frac{18}{2} = 9$ and $\beta_{8,3}(G) = 3$. **(2)** $G_1 = (V \cup \{s\}, E \cup F_1)$ obtained by Step I. Edges in F_1 are drawn as broken lines. Now G_1 satisfies (3.1) for $\ell = 8$, (3.2) and (3.3), and the edge $\hat{e} = (s, w)$ is extra. **(2-1)** $G_1' = (V \cup \{s\}, E \cup F_1')$ with $F_1' \subseteq F_1$ in Step II, satisfying (3.1) and (5.1). **(2-2)** $G_2' = (V \cup \{s\}, E \cup (F_1 - F_1') \cup F_2')$ in Step II, where F_2' is a set of edges obtained from splitting edges in F_1'. Now $G_2'[V]$ is $(8, 2)$-connected and every edge in $E_{G_2'}(s, V) - \hat{e}$ is κ-critical. **(2-3)** $G_2'' = (V \cup \{s\}, E \cup F_2)$ in Step II obtained from splitting edges in $F_1 - F_1'$ in G_2', which creates no self-loop. **(3)** $G_2 = (V, E \cup F_2)$ obtained by Step II. The G_2 satisfies $\lambda(G_2) \geq 8$ but has a disconnecting pair $S = \{v^*, v\}$. **(4)** $G_2^{(1)} = (V, E \cup F_2^{(1)})$ obtained from G_2 by an improving switching $\{(u_1, u_2), (w_1, w_2)\}/\{(u_1, w_1), (u_2, w_2)\}$ in Step III. Moreover, any switching is no longer improving in $G_2^{(1)}$. **(5)** $G_4 = (V, E \cup F_4)$ obtained by shifting $\{(u_1, u_3)\}/\{(v^*, u_3)\}$ in Step IV. This G_4 is $(8, 3)$-connected. $\square$

If $\kappa(G_2) \geq 3$ holds, then we are done, because $|F_2| = |F_1|/2 = \lceil \alpha_{\ell,3}(G)/2 \rceil$ attains the lower bound. Otherwise ($\kappa(G_2) = 2$), (3.2) and (3.3) imply the following properties to the family $\mathcal{D}(G_2)$ of minimal tight sets in G_2.

$$\begin{array}{l} T \cap V[F_2] \neq \emptyset \text{ for all } T \in \mathcal{D}(G_2), \\ |E_{G_2}(x) \cap F_2| \geq 2 \text{ for all } T = \{x\} \in \mathcal{D}(G_2) \text{ with } |\Gamma_G(x)| = 1, \\ |E_{G_2}(x) \cap F_2| \geq 3 \text{ for all } T = \{x\} \in \mathcal{D}(G_2) \text{ with } |\Gamma_G(x)| = 0. \end{array} \tag{3.4}$$

Step III (Switching edges): During this step, we try to make the current $(\ell, 2)$-connected $G_2 = (V, E \cup F_2)$ 3-vertex-connected by switching some edges in F_2 while preserving the ℓ-edge-connectivity. Some sufficient conditions are given below for G_2 to admit an improving switching such that at least one new pair of vertices in G_2 becomes locally 3-vertex-connected.

Let $F^*(G_2) = E^*(G_2) \cap F_2$. For a disconnecting pair S in G_2, let $F^a(S; G_2)$ be a set of edges in $F_2 - F^*(G_2)$ which are s-admissible with respect to S and $F^n(S; G_2)$ be a set of edges in $F_2 - F^*(G_2)$ which are not s-admissible with respect to S.

Property 3.3 [15, Property 3.2] *If G_2 has a disconnecting pair $S = \{v_1, v_2\}$ and edges $e_1 = (u_1, w_1), e_2 = (u_2, w_2) \in F_2$ such that one of the following* (a)–(d) *holds and $\kappa(G_2 - \{e_1, e_2\}) \geq 2$, then one of switchings $\{(u_1, u_2), (w_1, w_2)\}/\ \{e_1, e_2\}$ and $\{(u_1, w_2), (w_1, u_2)\}/\ \{e_1, e_2\}$ in F_2 is improving to G_2 and the number of S-components in G_2 decreases at least by one after the switching.*

(a) *For some distinct S-components T_1 and T_2 in G_2, $V[e_1] \subseteq T_1$ and $V[e_2] \subseteq T_2$ hold, and at least one of e_1 and e_2 is s-admissible with respect to S.*

(b) *For some disconnecting pair S' of G_2 with $S \cap S' = \emptyset$, there are an S'-component T_1 and an S-component T_2 such that $T_1 \cap T_2 = \emptyset$, $V[e_1] \subseteq T_1 \cup S'$, $V[e_2] \subseteq T_2 \cup S$ and $V[e_i] \cap T_i \neq \emptyset$ $(i = 1, 2)$.*

(c) *For some disconnecting pair $S' = \{v_1, v_3\}$ of G_2 with $v_3 \neq v_2$, there are an S'-component T_1 and an S-component T_2 such that $T_1 \cap T_2 = \emptyset$, $V[e_1] \subseteq T_1 \cup \{v_3\}$ and $V[e_2] \subseteq T_2 \cup \{v_2\}$.*

(d) *For some two distinct S-components T_1 and T_2 in G_2, it holds that $v_1 \in V[e_1] \subseteq T_1 \cup \{v_1\}$ and $v_2 \in V[e_2] \subseteq T_2 \cup \{v_2\}$.* □

However, if an initial multigraph G is not 2-vertex-connected (a situation not assumed in [15]), there may be an edge $e \in F_2$ such that $\kappa(G_2 - e) < 2$, and hence Property 3.3 may not be applied to G_2.

In this case, to apply the property repeatedly unless there is no edge pair e_1, e_2 satisfying any of (a) – (d), we need to show that there always exists a pair of two edges $e_1, e_2 \in F_2$ satisfying one of (a) – (d) and $\kappa(G_2 - \{e_1, e_2\}) \geq 2$.

Property 3.4 *For each tight set* T *in* G_2, $G_2[T \cup \Gamma_{G_2}(T)]$ *contains at least one edge* $e \in F_2 - F^*(G_2)$ *with* $T \cap V[e] \neq \emptyset$. □

Property 3.5 *Let* S *be a disconnecting pair in* G_2. *If two edges* $e_1, e_2 \in F_2 - F^*(G_2)$ *satisfy* $T_i \cap V[e_i] \neq \emptyset$, $i = 1, 2$, *for two distinct* S*-components* T_1 *and* T_2 *in* G_2, *then* $G_2 - \{e_1, e_2\}$ *is 2-vertex-connected.* □

From these properties, every tight set contains an edge in $F_2 - F^*(G_2)$ and the two edges $e_1, e_2 \in F_2 - F^*(G_2)$ satisfying one of the above conditions (a)–(d) always satisfy $\kappa(G_2 - \{e_1, e_2\}) \geq 2$. Therefore we can repeat executing an improving switching of pairs of edges in $F_2 - F^*(G_2)$ until none of the above conditions (a)–(d) holds in G_2. Then it is not difficult to see that (I) all disconnecting pairs S_i in the resulting graph G_2 contain one common vertex, say v^*; i.e.,

$$S_i = \{v^*, v_i\}, \; i = 1, \ldots, q. \tag{3.5}$$

In the following, let v^* satisfy $|E_{G_2}(v^*) \cap (F_2 - F^*(G_2))| \geq |E_{G_2}(v_1) \cap (F_2 - F^*(G_2))|$ in the case of $q = 1$.

We now consider another case in which an improving switching can be found.

Property 3.6 *Let* $S_i = \{v^*, v_i\}$ *be a disconnecting pair in* G_2. *If* G_2 *has three edges* $e_k = (u_k, w_k) \in F_2 - F^*(G_2)$ $(k = 1, 2, 3)$ *such that the following condition* (e) *holds, then one of the switchings* $\{(v^*,\, v_i), (w_1, u_3), (w_2, w_3)\}/\ \{e_1, e_2, e_3\}$ *and* $\{(v^*, v_i), (w_1, w_3), (w_2, u_3)\}/\{e_1, e_2, e_3\}$ *in* F_2 *is improving to* G_2 *and decreases the number of* S_i*-components in* G_2 *at least by one.*

(e) *For some two distinct* S_i*-components* T_1 *and* T_2, *it holds that* $u_1 = v^*$, $u_2 = v_i$, $\{w_1, w_2\} \subseteq T_1$ *and* e_3 *is contained in* $G_2[T_2]$. □

This is a generalization of the property shown in [15, Property 3.3] to the case where $\kappa(G_2 - \{e_1, e_2, e_3\})$ may be less than 2.

If none of the conditions (a)-(e) holds any longer, let $G_3 = (V, E \cup F_3)$ denote the resulting multigraph, where $F_3 = F_2$. If G_3 has no disconnecting pair (i.e., $\kappa(G_3) \geq 3$), then we halt by outputting F_3 as an optimal solution, since $|F_3| = \lceil \alpha_{\ell,3}(G)/2 \rceil$ attains our lower bound $\gamma_{\ell,3}(G)$. Otherwise we go to Step IV.

Step IV (Shifting edges): Then G_3 has the following properties.

Property 3.7 *Let* $S_i = \{v^*, v_i\}$, $i = 1, \ldots, q$, *denote all disconnecting pairs in* G_3.
(1) *For each* S_i, *any edge in* $F_3 - F^*(G_3)$ *incident to* v_i *is also incident to* v^*.
(2) *For each* S_i, *if* $F^a(S_i; G_3) \neq \emptyset$ *holds, then* $V[F^a(S_i; G_3)] \subseteq T_1$ *holds for exactly one* S_i*-component* T_1 *and all edges in* $F_3 - F^*(G_3)$ *that are contained in* S_i*-components other than* T_1 *are incident to* v^*.
(3) *For each* $e \in F^*(G_3)$, *it holds that* $e = (v_i, v_j)$ *for some* $i \neq j$, *or* $e = (t, v_i)$ *with* $\Gamma_{G_3}(t) = \{v^*, v_i\}$ *for some* i. □

For the common vertex v^* of (3.5), we apply the following shifting operation in F_3.

Property 3.8 [15, Property 3.5] *If $c_{G_3}(v^*) > \ell$ and $E_{G_3}(v^*) \cap (F_3 - F^*(G_3)) \neq \emptyset$ holds, then G_3 has an edge $(v^*, w) \in F_3$ satisfying at least one of the following properties* (i), (ii) *and* (iii). *In particular, if every edge $e \in F_3 - F^*(G_3)$ satisfies $v^* \in V[e]$, then at least one edge in $F_3 - F^*(G_3)$ satisfies* (i). *Moreover, each of such shifting decreases the number of S-components for some disconnecting pair S in G_3.*

(i) *For some vertex $u \in V - v^*$, shifting $(w, u)/(v^*, w)$ is improving to G_3.*

(ii) *For an edge $(u', w') \in F_3 - F^*(G_3)$ such that w and $\{u', w'\}$ are disconnected by some disconnecting pair in G_3 and $G_3 - \{(v^*, w), (u', w')\}$ is 2-vertex-connected, shifting $\{(u, u'), (w, w')\}/ \{(v^*, w), (u', w')\}$ is improving to G_3 for a vertex $u \in V - v^*$.*

(iii) *For an edge $(u', w') \in F_3 - F^*(G_3)$ such that w and $\{u', w'\}$ are disconnected by some disconnecting pair in G_3 and $G_3 - \{(v^*, w), (u', w')\}$ is 2-vertex-connected, shifting $\{(v^*, u'), (w, w')\}/ \{(v^*, w), (u', w')\}$ is improving to G_3.* ☐

From Property 3.7, $E_{G_3}(v^*) \cap F_3 \subseteq F_3 - F^*(G_3)$ follows. Property 3.4 says that for every tight set T in G_3, there is an edge $e \in F_3 - F^*(G_3)$ with $V[e] \cap T \neq \emptyset$. Furthermore, from Property 3.5, $\kappa(G_3 - \{(v^*, w), (u', w')\}) \geq 2$ holds in (ii) and (iii), since w and $\{u', w'\}$ are disconnected by a disconnecting pair in G_3. Therefore, we can apply Property 3.8 to G_3, even if an initial G is not 2-vertex-connected. The shifting in Property 3.8 reduces $|F_3 \cap E_{G_3}(v^*)|$ at least by one. Therefore we can repeat improving shifting and switching of edges in $F_3 - F^*(G_3)$ until $|F_3 \cap E_{G_3}(v^*)| = 0$ or $c_{G_3}(v^*) = \ell$ holds.

Property 3.9 *If G_3 has an edge in $F^*(G_3)$, then $c_{G_3}(v^*) \geq \ell + 1$ holds.* ☐

Property 3.10 *If $F^*(G_3) \neq \emptyset$ holds, then we can execute an improving switching or shifting of edges in $F_3 - F^*(G_3)$ by applying Properties 3.3, 3.6 or 3.8.* ☐

Let $G_4 = (V, E \cup F_4)$ denote the resulting multigraph $G_3 = (V, E \cup F_3)$ obtained by repeating an improving switching or shifting of edges in F_3 until none of conditions in Properties 3.3, 3.6 or 3.8 holds in G_3. It holds $F^*(G_4) = \emptyset$ from Property 3.10. Now we have $c_{G_4}(v^*) = \ell$ in G_4 whenever $E_{G_4}(v^*) \cap F_4 \neq \emptyset$; the number of edges in F_4 incident to S_i is equal to $\max\{\ell - c_G(v^*), \ell - c_G(v_i), 0\}$ for any disconnecting pair $S_i = \{v^*, v_i\}$.

For a disconnecting pair S_i in G_4, an edge $e \in F^n(S_i; G_4) - E_{G_4}[S_i]$ is called *dividing* with respect to S_i, if $E(G_4[X_i]) \cap (F^n(S_i; G_4) - E_{G_4}[S_i]) \neq \emptyset$ holds for $i = 1, 2$, where X_1 and X_2 are two S-components in $G_4 - e$ with $T_1 = X_1 \cup X_2$ and $E_{G_4}(X_1, X_2) = \{e\}$ for the S-component T_1 in G_4 with $V[e] \subseteq T_1$. Then we consider the following property.

Property 3.11 [15, Property 3.7] *Assume that $F^a(S; G_4) \neq \emptyset$ holds for each disconnecting pair S in G_4. Let T_1 be an S_1-component in G_4 containing an edge in $F^a(S_1; G_4)$ such that no S'-component $T' \subset T_1$ with $S' \neq S_1$ contains an edge in $F^a(S'; G_4)$, where S_1 and S' are* *disconnecting pairs in* G_4. *(Recall that all edges in* $F_4 - F^*(G_4)$ *that are contained in* $T_i \in \mathcal{C}(G_4 - S_1) - T_1$ *are incident to* v^*.*) If* $|F^a(S_1; G_4)| \geq \min\{\ell - 1, n - 2, c_G(v^*, T_1) + 1\}$ *holds, then there is a sequence of improving switchings of edges in* $F_4 - F^*(G_4)$ *which decreases the number of the* S_1*-components by one.* □

Property 3.12 *Assume that for some disconnecting pair* S_1 *in* G_4*, there is an edge in* $F^n(S_1; G_4)$ *which is dividing with respect to* S_1*. If* $F^a(S_1;\, G_4) \neq \emptyset$ *holds or the number of disconnecting pairs in* G_4 *is at least two, then there is a sequence of switching of edges in* $F_4 - F^*(G_4)$ *such that the resulting graph* G_4' *satisfies* $p(G_4' - S) \leq p(G_4 - S)$ *for each pair* S *of two vertices in* V*. Moreover,* G_4' *satisfies (a)* $p(G_4' - S') < p(G_4 - S')$ *for some disconnecting pair* S' *in* G_4 *or (b)* $F^a(S_1; G_4') = \emptyset$ *holds and* G_4' *has no edge in* $F^n(S_1; G_4')$ *dividing with respect to* S_1*.* □

It is not difficult to see that we apply Property 3.11 to G_4, since Property 3.11 uses only switching operations that change the degree of no vertex in V and Property 3.9 says that $c_{G_4}(v^*) = \ell$ maintains condition $F^*(G_4) = \emptyset$ (note that $F_4 \cap E_{G_4}(v^*) \neq \emptyset$ implies $c_{G_4}(v^*) = \ell$ in the cases of applying Property 3.11).

We now repeat switching and shifting edges in F_4 as long as none of conditions in Properties 3.3, 3.6, 3.8, 3.11 or 3.12. Let $G_5 = (V, E \cup F_5)$ denote the resulting graph. If G_5 has no disconnecting pair, then we are done, since $|F_5| = \lceil \alpha_{\ell,3}(G)/2 \rceil$ implies that G_5 is optimally augmented. Otherwise, go to Step V. Such G_5 is characterized as follows.

Property 3.13 *Let* $S_i = \{v^*, v_i\}$ $(i = 1, \ldots, q)$ *denote all the disconnecting pairs in* G_5*, where* $|E_{G_5}(v^*) \cap F_5| \geq |E_{G_5}(v_1) \cap F_5|$ *is assumed if* $q = 1$*. Then we have* $F^*(G_5) = \emptyset$ *and any edge in* F_5 *incident to* v_i *is also incident to* v^**. Moreover,* G_5 *satisfies one of the following properties:*

(i) $q = 1$ *and* $|E_{G_5}(v^*) \cap F_5| = \max\{0, \ell - c_G(v^*)\}$ *hold and none of the edges in* F_5 *is s-admissible with respect to* S_1*.*

(ii) $c_{G_5}(v^*) = \ell$ *and* $|E_{G_5}(v^*) \cap F_5| \geq 1$ *hold, and each disconnecting pair* S_i *satisfies the following: None of edges in* $F^n(S_i; G_5)$ *is dividing with respect to* S_i*. If* $F^a(S_i;\, G_5) \neq \emptyset$ *holds, then all edges in* F_5 *which are not incident to* v^* *are contained in exactly one* S_i*-component* T*. Furthermore, there is a disconnecting pair* S *such that at most* $\ell - 2$ *edges in* F_5 *are s-admissible with respect to* S*.* □

Step V (Edge augmentation):

Property 3.14 *If* G_5 *satisfies (i) of Property 3.13, then* $p(G_5 - S_1) = p(G - S_1) + \max\{\ell - c_G(v^*), \ell - c_G(v_1), 0\} - |F_5|$*, where* $S_1 = \{v^*, v_1\}$ *is a disconnecting pair in* G_5*.* □

From this and the definition of $\beta_{\ell,3}(G)$, we obtain $p(G_5 - S_1) - 1 = p(G - S_1) - 1 + \max\{\ell - c_G(v^*), \ell - c_G(v_1), 0\} - |F_5| \leq \beta_{\ell,3}(G) - |F_5|$. Let $T_1, \ldots, T_r$ be all the S_1-components in G_5, where $r = p(G_5 - S_1)$. Add to G_5 a set of $r - 1$ new edges $F_6 = \{(x_i, x_{i+1}) \mid i = 1, \ldots, r-1\}$, where x_i is an arbitrary vertex in T_i. This makes G_5 3-vertex-connected. Note that $|F_5| + |F_6| = |F_5| + p(G_5 - S_1) - 1 \leq \beta_{\ell,3}(G)$ implies that $F_5 \cup F_6$ is an optimal solution.

Now we consider the case in which G_5 satisfies (ii) of Property 3.13 (note that if $c_G(v) \geq \ell$ holds for all $v \in V$ originally, then case (ii) in Property 3.13 cannot occur). Then an upper bound on $opt_{\ell,3}(G)$ is given by

$$opt_{\ell,3}(G) \leq |F_5| + |\mathcal{D}(G_5)| - 1, \tag{3.6}$$

because G_5 (which is already ℓ-edge-connected) can be augmented to a 3-vertex-connected graph by another set of $|\mathcal{D}(G_5)| - 1$ new edges.

By (3.4), any minimal tight set in G_5 contains an end vertex of an edge in F_5. As any two minimal tight sets in $\mathcal{D}(G_5)$ are pairwise disjoint in G (by Lemma 2.1), we have $|\mathcal{D}(G_5)| \leq |F_5| = \lceil \alpha_{\ell,3}(G)/2 \rceil$ and hence $opt_{\ell,3}(G) \leq 2\lceil \alpha_{\ell,3}(G)/2 \rceil - 1$ by (3.6). Furthermore we can show that $|F_5| = O(\ell)$ holds.

Property 3.15 *If G_5 satisfies* (ii) *of Property 3.13, then we have $c_G(v^*) < \ell$, $|F_5| = \lceil \alpha_{\ell,3}(G)/2 \rceil \leq 2\ell$, $|F_5| + |\mathcal{D}(G_5)| - 1 \leq 5\ell/2 - 1$, $|\mathcal{D}(G_5)| \leq \ell/2$ and $opt_{\ell,3}(G) \leq \min\{5\ell/2 - 1, 2\lceil \alpha_{\ell,3}(G)/2 \rceil - 1\}$.* □

We can show that adding a set F_6' of new edges to make G_5 3-vertex-connected, where F_6' satisfies $|F_6'| \leq \min\{\lceil \alpha_{\ell,3}(G)/2 \rceil - 1, \ell/2 - 1\}$, we obtain an approximate solution $F' = F_5 \cup F_6'$ with $|F'| \leq \min\{2opt_{\ell,3}(G) - 1, 5\ell/2 - 1\}$ and $|F'| - opt_{\ell,3}(G) \leq \ell/2 - 1$. An optimal solution F can then be found by inspecting all possible choices of F with $|F| \leq 5\ell/2 - 1$. □

4 Justification of Step I

We observe that the edge set F_1 in Step I in algorithm EV-AUGMENT3* can be found in polynomial time by using the g-polymatroid intersection theorem of Frank [5]. In Appendix B, we present a graph theoretical algorithm which finds F_1 in $O(n^2 m + n^3 \log n)$ time.

Let V be a finite ground set and let $q : 2^V \to Z \cup \{-\infty\}$ be a function with $q(\emptyset) = 0$, where Z denotes a set of integers. Define $C(q) = \{z \in R^V | z \geq 0, z(X) \geq q(X) \text{ for every } X \subset V\}$, where R denotes the set of real numbers and $z(X) = \sum_{i \in X} z(i)$. Let $q^* : 2^V \to Z \cup \{-\infty\}$ denote the function such that

$$q^*(X) = \max\left\{\sum_{i=1}^{t} q(X_i) | \{X_1, \ldots, X_t\} \text{ is a subpartition of } X\right\}$$

for all $X \subseteq V$. Then the following claim holds.

Claim 4.1 *$C(q) = C(q^*)$ holds.*

Proof: It is clear that every $z \in C(q^*)$ satisfies $z \in C(q)$, from which $C(q^*) \subseteq C(q)$. For every $z \in C(q)$, we have $z(X) \geq \max\{\sum_{i=1}^{t} q(X_i) | \{X_1, \ldots, X_t\}$ is a subpartition of $X\}$ for every

$X \subseteq V$, since $z \in C(q)$ implies $z(X_i) \geq q(X_i)$ for each X_i and we have $z(X - \sum_{i=1}^{t} X_i) \geq 0$. Hence $C(q) \subseteq C(q^*)$ holds. □

We call q *fully supermodular* if $q(X)+q(Y) \leq q(X \cap Y)+q(X \cup Y)$ holds for every $X, Y \subseteq V$. If $q(Y) \leq q(X)$ holds for every $X, Y \subseteq V$ with $Y \subseteq X$, then q is called *monotone*. Note that q^* is always monotone. For a fully supermodular and monotone function q, the set $C(q)$ is called the *contrapolymatroid of* q.

For a graph $G = (V, E)$ and a nonnegative integer ℓ, let $q_1(X) = \ell - c_G(X)$ for all cuts $X \subset V$ and $q_1(\emptyset) = q_1(V) = 0$. For a graph $G = (V, E)$, let $q_2(X) = 3 - |\Gamma_G(X)|$ if a set $X \subseteq V$ satisfies $V - X - \Gamma_G(X) \neq \emptyset$ and $X \neq \emptyset$, and $q_2(X) = 0$ otherwise. Then, by the following property, we see that the problem of finding F_1 in Step I can be solved by finding an integer valued vector $z \in C(q_1) \cap C(q_2)$ ($= C(q_1^*) \cap C(q_2^*)$ by Claim 4.1) such that $z(V)$ is the minimum.

Property 4.1 *A graph $G' = (V \cup \{s\}, E \cup F')$ satisfies* (3.1) (*resp.,* (3.2) *and* (3.3)) *if and only if a vector $z \in Z^V$ such that $z(v) = c_{G'}(v)$ for all $v \in V$ satisfies $z \in C(q_1)$ (resp., $z \in C(q_2)$).*

Proof: The former property is clear. Here we show the latter property. For this, it suffices to show that G' satisfies (3.2) and (3.3) if and only if G' satisfies

$$c_{G'}(s, X) \geq q_2(X) \text{ for all } X \subseteq V. \tag{4.1}$$

It is clear that G' satisfying (3.2) and (3.3) satisfies (4.1). We show that G' satisfying (4.1) satisfies (3.2) and (3.3). Clearly, if G' satisfies (4.1), then G' satisfies (3.3). Let $X \subset V$ be a cut with $V - X - \Gamma_G(X) \neq \emptyset$ and $|X| + |\Gamma_G(X)| \geq 3$. If $|\Gamma_G(X)| \geq 2$ holds, then G' clearly satisfies (3.2). If $|\Gamma_G(X)| = 1$ holds, then $X - v$ satisfies $c_{G'}(s, X - v) \geq q(X - v) = 1$ for every $v \in X$, from which (3.2) holds in G'. If $|\Gamma_G(X)| = 0$ holds, then $X - \{u, v\}$ satisfies $c_{G'}(s, X - \{u, v\}) \geq q(X - \{u, v\}) = 1$ for every pair $\{u, v\} \subset X$, from which (3.2) holds in G'.□

Now we observe the following lemmas about $C(q_1)$ and $C(q_2)$.

Lemma 4.1 [6] *q_1^* is a monotone and fully supermodular function. That is, $C(q_1)$ is a contrapolymatroid.* □

Lemma 4.2 *q_2^* is a monotone and fully supermodular function. That is, $C(q_2)$ is a contrapolymatroid.*

Proof: Clearly, p is monotone. So, it suffices to show that p is fully supermodular.

Claim 4.2 *Let X be a set of vertices in V and $G' = (V \cup \{s\}, E \cup F')$ be a graph satisfying* (4.1) *such that $F' = E_{G'}(s)$ holds, $c_{G'}(s, v) = 3$ holds for every $v \in V - X$ and the deletion of any one edge $(s, v) \in F'$ with $v \in X$ violates (4.1). Then we have $p(X) = c_{G'}(s, X)$.*

Proof: See Appendix A. □

Let X and Y be two sets of vertices in V and let $G^1 = (V \cup \{s\}, E \cup F^1)$ with $F^1 = E_{G^1}(s)$ and $c_{G^1}(s, v) = 3$ for all $v \in V$. Note that G^1 satisfies (4.1). Let G^2 (resp., G^3, G^4) be a graph obtained from deleting edges in $E_{G^1}(s, X \cap Y)$ (resp., $E_{G^2}(s, X-Y)$, $E_{G^3}(s, Y-X)$) so that G^2 (resp., G_3, G_4) preserves (4.1) and the deletion of any one edge in $E_{G^2}(s, X \cap Y)$ (resp., $E_{G^3}(s, X-Y)$, $E_{G^4}(s, Y-X)$) violates (4.1). Claim 4.2 implies $p(X \cap Y) = c_{G^2}(s, X \cap Y)$, $p(X) = c_{G^3}(s, X)$ and $p(X \cup Y) = c_{G^4}(s, X \cup Y)$. We see $p(Y) \leq c_{G^4}(s, Y)$ from the definition of $p(Y)$ and the property that G^4 satisfies (4.1). From the construction, we have $c_{G^4}(s, X \cap Y) = c_{G^2}(s, X \cap Y)$ and $c_{G^4}(s, X-Y) = c_{G^3}(s, X-Y)$. Hence, we have $p(X \cap Y) + p(X \cup Y) = c_{G^4}(s, X \cap Y) + c_{G^4}(s, X \cup Y)$ and $p(X) + p(Y) \leq c_{G^4}(s, X) + c_{G^4}(s, Y)$. Therefore $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ holds. ☐

By Lemmas 4.1 and 4.2 and Frank's [5] "g-polymatroid intersection theorem," the system $\mathcal{S}(q_1, q_2) = \{z \in R^V | z \geq 0, z(X) \geq \max\{q_1(X), q_2(X)\}$ for every $X \subseteq V\} = \{z \in C(q_1) \cap C(q_2)\}$ is a "submodular flow system" and hence an integer valued vector $z^* \in \mathcal{S}(q_1, q_2)$ minimizing $z(V)$ can be found in polynomial time [2]. From these properties, we observe $z^*(V) = \alpha_{\ell,3}(G)$. Thus, we see that the set F_1 can be found in polynomial time.

5 Justification for Step II

We show how to obtain an $(\ell, 2)$-connected G_2 from a complete and loopless splitting at s in G_1. A vertex v is called a *disconnecting vertex* in G if $S = \{v\}$ is a disconnecting set in G.

Algorithm SPLIT

Input: An undirected multigraph $G_1 = (V \cup \{s\}, E \cup F_1)$ with $|V| \geq 4$ and an integer $\ell \geq 4$ such that $|F_1|$ is even and G_1 satisfies (3.1) – (3.3).

Output: An undirected multigraph $G_2 = (V, E \cup F_2)$ with $|F_2| = |F_1|/2$ such that G_2 is $(\ell, 2)$-connected.

1. Let $F_1' \subseteq F_1$ be a set of edges such that $G_1' = (V \cup \{s\}, E \cup F_1')$ satisfies (3.1) and the following (5.1), and F_1' be minimal subject to this property.

$$\begin{aligned} |\Gamma_G(X)| + |\Gamma_{G_1'}(s) \cap X| \geq 2 \qquad & \text{for all cuts } X \subset V \text{ with } V - X - \Gamma_G(X) \neq \emptyset \\ & \text{ and } |\Gamma_G(X)| + |X| \geq 2, \\ c_{G_1'}(s, x) \geq 2 \qquad & \text{for all } X = \{x\} \text{ with } \Gamma_G(X) = \emptyset, \end{aligned} \tag{5.1}$$

where if $c_{G_1'}(s)$ is odd, then we choose one edge $e^* \in F_1 - F_1'$, and update $G_1' := G_1' + e^*$ and $F_1' := F_1' + e^*$.

It is shown [14] that by a complete splitting at s in G_1', which does not create a self-loop, we can obtain a multigraph $G_2' = (V \cup \{s\}, E \cup (F_1 - F_1') \cup F_2')$ satisfying one of the following conditions, where F_2' is the set of edges obtained from splitting edges in F_1'.

(i) $G_2'[V]$ is $(\ell, 2)$-connected.

(ii) $G_2'[V]$ is $(\ell$,1)-connected and has exactly one disconnecting vertex v_0. At most one edge in F_2' is incident to v_0, and each edge $e \in F_2'$ which is not incident to v_0 satisfies $p(G_2'[V] - v_0) = p((G_2'[V] - e) - v_0) - 1$.

If G_2' satisfies (ii), go to step 2; otherwise go to step 3.

2. Let T_i, $i = 1, \ldots, r'$, denote all v_0-components in $G_2'[V]$. If $G_2'[V]$ has the edge $e = (v_0, t_1) \in F_2'$ incident to v_0, then assume $t_1 \in T_1$ without loss of generality.

Property 5.1 *We have* $c_{G_2'}(s, T_1) \geq 1$ *and* $c_{G_2'}(s, T_i) \geq 2$ *for* $i = 2, \ldots, r'$. □

Let $G_2'' = (V \cup \{s\}, E \cup (F - F_1' - F_1'') \cup F_2' \cup F_2'')$ be the multigraph resulting from a sequence of splitting pairs of edges (s, a_i) and (s, b_{i+1}), $i = 1, \ldots, r' - 1$, where $a_1 \in T_1 \cap \Gamma_{G_2'}(s)$, $a_i, b_i \in T_i \cap \Gamma_{G_2'}(s)$ for $i = 2, \ldots, r'$, $F_1'' = \{\{(s, a_i), (s, b_{i+1})\} \mid i = 1, \ldots, r' - 1\}$, and $F_2'' = \{(a_i, b_{i+1}) \mid i = 1, \ldots, r' - 1\}$. This $G_2''[V]$ is now $(\ell, 2)$-connected. Let $G_2' := G_2''$, $F_1' := F_1' \cup F_1''$, and $F_2' := F_2' \cup F_2''$, and go to 3.

3. Now $G_2'[V]$ is $(\ell, 2)$-connected and $c_{G_2'}(s)$ is even. If $c_{G_2'}(s) = 0$ holds, then output $G_2 := G_2' - s$.

Otherwise G_2' has the following property.

Property 5.2 $c_{G_2'}(s)$ *is even, and* $c_{G_2' - \hat{e}}(s, u) \leq 2$ *holds for every edge* $e = (s, u) \in F_1 - F_1' - \hat{e} = E_{G_2'}(s, V) - \hat{e}$. □

Compute a multigraph G_2'' from G_2' by a complete and loopless splitting at s in G_2' after at most one shifting of the extra edge $\hat{e}$ (chosen in Step I), and output $G_2 := G_2'' - s$. Such G_2'' can be obtained as follows.

If $|\Gamma_{G_2'}(s)| \geq 3$ holds, then G_2'' can be obtained by repeatedly splitting two edges (s, u_1) and (s, u_2) with $c_{G_2'}(s, u_1) = \max\{c_{G_2'}(s, u) \mid u \in V\}$ and $c_{G_2'}(s, u_2) = \max\{c_{G_2'}(s, u) \mid u \in V - u_1\}$, because at most one vertex $w \in V$ satisfies $c_{G_2'}(s, w) = 3$ and $\max\{c_{G_2'}(s, u) \mid u \in V - w\} \leq 2$ holds.

If $|\Gamma_{G_2'}(s)| = 2$ holds where $\Gamma_{G_2'}(s) = \{u_1, u_2\}$, then either $c_{G_2'}(s, u_1) = c_{G_1'}(s, u_2) \leq 2$, or $c_{G_2'}(s, u_1) = 3$ and $c_{G_1'}(s, u_2) = 1$. The first case is easy and we consider the second case. From Property 5.2, one edge in $E_{G_1'}(s, u_1)$ is the extra edge $\hat{e}$. Then we can obtain a complete and loopless splitting at s in G_2' by shifting $\hat{e}$ to (s, u_3) with $u_3 \neq u_1$.

If $|\Gamma_{G_2'}(s)| = 1$ holds where $\Gamma_{G_2'}(s) = \{u_1\}$, then $c_{G_2'}(s, u_1) = 2$ holds by Property 5.2. If one edge in $E_{G_1'}(s, u_1)$ is the extra edge $\hat{e}$, then we can obtain a complete splitting at s in G_2' which creates no self-loop by shifting $\hat{e}$ to (s, u_2) with $u_2 \neq u_1$. Otherwise we see $|\Gamma_G(u_1)| = 1$ and $V - u_1 - \Gamma_G(u_1) \neq \emptyset$, since $G_2' - e$ violates (3.2) or (3.3) for each $e \in E_{G_2'}(s)$. Moreover, every edge in $(F_1 - F_1') \cup F_2'$ incident to u_1 is in $E_{G_2'}(s, u_1)$ by $c_{G_2'}(s, u_1) = 2$. This implies that $G_2'[V]$ has a disconnecting vertex v with $\Gamma_G(u_1) = \{v\}$, which contradicts that $G_2'[V]$ is $(\ell, 2)$-connected. □

Here, in order to see the correctness of the algorithm SPLIT, we prove the following two Properties 5.1 and 5.2.

Proof of Property 5.1: For each T_i, $i = 2, \ldots, r'$, every edge $e \in F'_2 \cap E(G'_2[T_i])$ satisfies $p(G'_2[V] - v_0) = p((G'_2[V] - e) - v_0) - 1$ and hence there are at least two v_0-components $T_{i1}, T_{i2} \subseteq T_i$ in $G'_2[V] - (F'_2 \cap E(G'_2[T_i]))$ such that $|V(F'_2) \cap T_{i1}| = |V(F'_2) \cap T_{i2}| = 1$ holds. Now $c_{G_1}(s, T_{i1}) \geq 2$ and $c_{G_1}(s, T_{i2}) \geq 2$ hold by (3.2), (3.3), $|\Gamma_G(T_{i1})| \leq 1$ and $|\Gamma_G(T_{i2})| \leq 1$. Hence we have $c_{G'_2}(s, T_{i1}) \geq 1$ and $c_{G'_2}(s, T_{i2}) \geq 1$. Therefore $c_{G'_2}(s, T_i) \geq c_{G'_2}(s, T_{i1} \cup T_{i2}) \geq 2$ holds. Considering that G'_2 may have one edge $e = (v_0, t_1) \in F'_2$ with $t_1 \in T_1$, we see $c_{G'_2}(s, T_1) \geq 1$ in the same manner. $\Box$

Proof of Property 5.2: For each edge $e = (s, u) \in F_1 - F'_1 - \hat{e} = E_{G'_2}(s, V) - \hat{e}$, $G'_2 - \hat{e} - e$ violates (3.2) or (3.3) from the minimality of F_1. If $c_{G'_2 - \hat{e}}(s, u) = 3$, $\Gamma_G(u) = \emptyset$ and $V - u \neq \emptyset$ hold, then $G'_2 - \hat{e} - e$ violates (3.2) or (3.3), but this is impossible since $c_{G'_2 - \hat{e}}(u) = c_{G'_2 - \hat{e}}(s, u) \geq 4$ holds from (3.1) and $\ell \geq 4$. Hence $c_{G'_2 - \hat{e}}(s, u) \leq 2$ holds. $\Box$

The complexity of Step II depends on that of the algorithm EV-AUGMENT [14], and hence the $(\ell, 2)$-connected graph G_2 can be computed in $O((nm + n^2 \log n) \log n)$ time.

6 Justification for Step III

To show that Step III works correctly, we prove Properties 3.4 – 3.6. We start with the following lemma and claims.

Lemma 6.1 [15, Lemma 2.3] *Let $S = \{v_1, v_2\}$ be a disconnecting pair and T be an S-component in a 2-vertex-connected graph $G = (V, E)$. If G has an edge (v_1, v_2) in E, then the induced subgraph $G[T \cup S]$ is also 2-vertex-connected.* $\Box$

Claim 6.1 *For each cut T with $|\Gamma_{G_2}(T)| = 2$ and $V - T - \Gamma_{G_2}(T) \neq \emptyset$, if G_2 has an edge $e \in F_2$ with $T \cap V[e] \neq \emptyset$ such that $G_2 - e$ is not 2-vertex-connected, then $G_2 - e$ has a disconnecting vertex $v \in T \cup \Gamma_{G_2}(T)$ and has a v-component $T' \subseteq T$.*

Proof: Assume that $G_2 - e$ is not 2-vertex-connected. By $\kappa(G_2) \geq 2$, $\kappa(G_2 - e) \geq 1$ holds, and hence $G_2 - e$ has a disconnecting vertex v. Let $\Gamma_{G_2}(T) = \{v_1, v_2\}$ with $v_1 \notin V[e]$ without loss of generality and $e = (u, w)$. Now we assume that all disconnecting vertices v in $G_2 - e$ satisfies $v \in V - (T \cup \Gamma_{G_2}(T))$ and derive a contradiction. Since u and w are disconnected by v in $G_2 - e$, every path connecting u and w in $G_2 - e$ always contains $v \in V - (T \cup \Gamma_{G_2}(T))$ and hence always contains both v_1 and v_2 from $V[e] \cap T \neq \emptyset$. Therefore v_1 is also a disconnecting vertex in $G_2 - e$ from $v_1 \notin V[e]$, contradicting that all disconnecting vertices v in $G_2 - e$ satisfies $v \in V - (T \cup \Gamma_{G_2}(T))$.

Furthermore, we can easily see that $G_2 - e$ has a v-component $T' \subseteq T$, since $G_2[V - T] - e$ is still connected and $v \in T \cup \Gamma_{G_2}(T)$ holds, implying that the v-component in $G_2 - e$ containing a vertex in $V - T$ contains all vertices in $V - T$. $\Box$

Claim 6.2 [15, Claim 6.2] *Let $S = \{v_1, v_2\}$ be a disconnecting pair in an $(\ell, 2)$-connected graph $G = (V, E)$ with $\ell \geq 2$. Let $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ be two edges in E with $V[e_1] \cap V[e_2] = \emptyset$, such that $\{u_1, w_1\} \subseteq T_1 \cup \{v_1\}$ and $\{u_2, w_2\} \subseteq T_2 \cup \{v_2\}$ hold for two S-components T_1 and T_2 in G. Assume that $G - \{e_1, e_2\}$ is 2-vertex-connected. Then switching $\{(u_1, u_2), (w_1, w_2)\}/\{e_1, e_2\}$ or $\{(u_1, w_2), (u_2, w_1)\}/\{e_1, e_2\}$ preserves the ℓ-edge-connectivity of G.* □

Proof of Property 3.4: Let $T_1 \subset T$ be a cut with $|\Gamma_{G_2}(T_1)| = 2$ and $V - T_1 - \Gamma_{G_2}(T_1) \neq \emptyset$, such that no $T' \subset T_1$ satisfies $|\Gamma_{G_2}(T')| = 2$ and $V - T' - \Gamma_{G_2}(T') \neq \emptyset$. Let $\Gamma_{G_2}(T_1) = \{v_1, v_2\}$. By (3.4), G_2 has an edge $e = (u, w) \in F_2$ with $u \in T_1$ without loss of generality. If $G_2 - e$ is not 2-vertex-connected, then there is a $T_2 \subseteq T_1$ with $u \in T_2$ without loss of generality such that $\Gamma_{G_2 - e}(T_2) = \{v'\}$ and $v' \in T_1 \cup \{v_1, v_2\}$ hold from Claim 6.1. We claim $v' \in \{v_1, v_2\}$, since otherwise $\Gamma_{G_2 - e}(T_2) = \{v'\}$ with $v' \in T_1$ imply that $\Gamma_{G_2}(T_2) = \{v', w\}$ and $T_2 \subseteq T_1 - v'$, contradicting the minimality of T_1. Let $v' = v_1$ without loss of generality. If $|T_2| \geq 2$ holds, then $\Gamma_{G_2}(T_2 - u) = \{v_1, u\}$ holds, contradicting the minimality of T_1. So $|T_2| = 1$ holds, i.e., $T_2 = \{u\}$ and $\Gamma_{G_2}(u) = \{v_1, w\}$. From the minimality of T_1, $T_2 = T_1$ and $w = v_2$. From 4-edge-connectivity of G_2 and $c_{G_2}(u, w) = 1$, $c_{G_2}(u, v_1) \geq 3$ holds. Since $\Gamma_G(u) = \{v_1\}$ or $\emptyset$, the number of edges in F_2 incident to u is at least two from (3.4), from which there is an edge $e' = (u, v_1) \in F_2$. Now $G_2 - e'$ is still 2-vertex-connected from $c_{G_2}(u, v_1) \geq 2$. Therefore this e' proves this property. □

Proof of Property 3.5: Property 3.4 says that for each disconnecting pair S in G_2, there are two edges $e_1, e_2 \in F_2$ such that $T_i \cap V[e_i] \neq \emptyset$, $i = 1, 2$ for two distinct T_1 and T_2 are distinct and both $G_2 - e_1$ and $G_2 - e_2$ are still 2-vertex-connected. Now we assume that $G_2 - e_1$ is still 2-vertex-connected but $G_2 - e_1 - e_2$ is not 2-vertex-connected, and we derive a contradiction. Let $e_2 = (u_2, w_2)$ and $S = \{v_1, v_2\}$. Claim 6.1 says that $G_2 - e_1 - e_2$ has a disconnecting vertex $v \in T_2 \cup S$ and a v-component $T_2' \subseteq T_2$. Now T_2' is also a v-component in $G_2 - e_2$ from $v \in T_2 \cup S$ and $T_2' \cap (T_1 \cup S) = \emptyset$, contradicting that $G_2 - e_2$ is 2-vertex-connected. □

Proof of Property 3.6: (1) We first show that at least one of switchings $\{(v^*, v_i), (w_1, u_3), (w_2, w_3)\}/\{e_1, e_2, e_3\}$ and $\{(v^*, v_i), (w_1, w_3), (w_2, u_3)\}/\{e_1, e_2, e_3\}$ preserves the ℓ-edge-connectivity of G_2. Since $G_2 - e_1 - e_3$ remains 2-vertex-connected from Property 3.5, by applying Claim 6.2, the resulting graph from at least one of switching $\{(v^*, u_3), (w_1, w_3)\}/ \{e_1, e_3\}$ or $\{(v^*, w_3), (w_1, u_3)\}/ \{e_1, e_3\}$ is ℓ-edge-connected. We next show that switching $\{(u_3, w_2),\ (v^*, v_i)\}/\{(v^*, u_3), e_2\}$ (resp., $\{(w_3, w_2), (v^*, v_i)\}/\{(v^*, w_3), e_2\}$) preserves the ℓ-edge-connectivity in the former case (resp., in the latter case). We prove only the former case, since the latter is similar. Let $G_2' := G_2 +\ \{(v^*, u_3), (w_1, w_3)\} -\ \{e_1, e_3\}$ and let G_2'' denotes the graph obtained from G_2' by switching $\{(u_3, w_2),\ (v^*, v_i)\}/\{(v^*, u_3), e_2\}$. If a cut X does not separate $\{v^*, v_i\}$ and $\{u_3, w_2\}$, then $c_{G_2''}(X) \geq c_{G_2'}(X) \geq \ell$ holds. We consider a cut X with $\{v^*, v_i\} \subseteq X$ and $\{u_3, w_2\} \subseteq V - X$. Note that $c_{G_2''}(X) = c_{G_2'}(X) - 2$ holds. If $w_1 \in X$ or $w_3 \in X$ holds, then $c_{G_2'}(X) = c_{G_2'}(V - X) \geq 2\ell \geq \ell + 2$ since $G_2'[V - X]$ is not connected. Hence $c_{G_2''}(X) = c_{G_2'}(X) - 2 \geq \ell$ holds. If $w_1 \notin X$ and $w_3 \notin X$ hold, then $c_{G_2'}(X) = c_{G_2}(X)$ holds by construction of G_2'. Since $G_2[V - X]$ is not connected, we have $c_{G_2}(X) = c_{G_2}(V - X) \geq 2\ell \geq \ell + 2$, and hence $c_{G_2''}(X) = c_{G_2'}(X) - 2 = c_{G_2}(X) - 2$

$\geq \ell$. Therefore $c_{G_2''}(X) \geq \ell$ holds for all cuts X.

(2) We show that at least one of switchings $\{(v^*, v_i), (w_1, u_3), (w_2, w_3)\}/\{e_1, e_2, e_3\}$ and $\{(v^*, v_i), (w_1, w_3), (w_2, u_3)\}/\{e_1, e_2, e_3\}$ satisfies $\mathcal{P}_3(G_2') \supseteq \mathcal{P}_3(G_2)$, where G_2' denotes the graph obtained from such switchings. If $G_2 - e_1 - e_2 - e_3$ is still 2-vertex-connected, then this property is equivalent to [15, Property 3.3]. Hence we assume that $G_2 - e_1 - e_2 - e_3$ is not 2-vertex-connected.

First we show that $G_2 - e_1 - e_2 - e_3$ has a disconnecting vertex $v \in T_1$ and a v-component $T_1' \subseteq T_1$ with $\{w_1, w_2\} \subseteq T_1'$. From Property 3.5, both $G_2 - e_1 - e_3$ and $G_2 - e_2 - e_3$ are still 2-vertex-connected from $\{e_1, e_2, e_3\} \subseteq F_2 - F^*(G_2)$. Hence $(G_2 - e_1 - e_3) - e_2$ is not 2-vertex-connected and has a disconnecting vertex $v \in T_1 \cup \{v^*\}$ and a v-component $T_1' \subseteq T_1$ from Claim 6.1 (note that $v \neq v_i$ holds since v disconnects vertices w_2 and v_i in $G_2 - e_1 - e_2 - e_3$). We here claim that v^* is not a disconnecting vertex in $G_2 - e_1 - e_2 - e_3$. Otherwise T_1 is a v^*-component in $G_2 - e_1 - e_2 - e_3$ since $G_2[T_1] - e_1 - e_2 - e_3 = G_2[T_1]$ is connected. Clearly, T_1 is a v^*-component in $G_2 - e_2 - e_3$, contradicting Property 3.5.

From above, $G_2 - e_1 - e_2 - e_3$ has a disconnecting vertex $v \in T_1$ and a v-component $T_1' \subseteq T_1$. Since both $G_2 - e_1 - e_3$ and $G_2 - e_2 - e_3$ are 2-vertex-connected, $\{w_1, w_2\} \subseteq T_1'$ holds.

Next we show $\mathcal{P}_3(G_2') \supseteq \mathcal{P}_3(G_2)$. Without loss of generality, let $G_2' := G_2 - \{e_1, e_2, e_3\} \cup \{(v^*, v_i), (w_1, u_3), (w_2, w_3)\}$ be ℓ-edge-connected. For this, we show (i) $\kappa_{G_2' \cup \{e_2, e_3\}}(v^*, w_1) \geq 3$, (ii) $\kappa_{G_2' \cup \{e_3\}}(v_i, w_2) \geq 3$, and (iii) $\kappa_{G_2'}(u_3, w_3) \geq 3$.

We first show (i). Since G_2' has an edge (v^*, v_i), $G_2'[T_1 \cup S_i]$ has two internally disjoint paths P_1 and P_2 between v^* and w_1 by Lemma 6.1 and $\kappa(G_2' - e_1) \geq 2$. There also exists a path P_3 between v^* and u_3 in $G_2' \cup \{e_2, e_3\} - (T_1 \cup \{v_i\})$ since T_2 is an S_i-component. By construction, all of P_1, P_2 and $P_3 \cup (u_3, w_1)$ are paths between v^* and w_1, which are internally disjoint. Therefore $\kappa_{G_2' \cup \{e_2, e_3\}}(v^*, w_1) \geq 3$.

We show (ii). We first claim that $(G_2' \cup \{e_3\})[T_1 \cup \{v_i\}]$ has two disjoint paths P_1 and P_2 such that P_1 (resp., P_2) is a path between w_2 and v_i (resp., w_1). Now $\kappa(G_2 - e_2) \geq 2$ implies that $(G_2' \cup \{e_1, e_3\})[T_1 \cup S_i]$ has two disjoint paths P_1 and P_1' such that P_1 (resp., P_1') is a path between w_2 and v_i (resp., v^*). But we see that P_1' always contains the edge $e_1 = (w_1, v^*)$, since w_2 and $\{v^*, v_i\}$ are disconnected by $v \in T_1$ in $G_2 - e_1 - e_2$. Then such paths P_1 and $P_2 := P_1' - e_1$ prove this claim.

Since G_2' has an edge (v^*, v_i) and $G_2 \cup \{(v^*, v_i)\}$ is 2-vertex-connected, $(G_2' \cup \{e_3\})[T_2 \cup S_i]$ $= (G_2 \cup \{(v^*, v_i)\})[T_2 \cup S_i]$ is 2-vertex-connected by Lemma 6.1. Hence $(G_2' \cup \{e_3\})[T_2 \cup S_i]$ has two internally disjoint paths P_3 and P_4 such that P_3 (resp., P_4) is between u_3 (resp., w_3) and v_i, since otherwise $(G_2' \cup \{e_3\})[T_2 \cup S_i]$ has a disconnecting vertex disconnecting $\{u_3, w_3\}$ and v_i, contradicting that $(G_2' \cup \{e_3\})[T_2 \cup S_i]$ is 2-vertex-connected.

From construction, all of P_1, $P_2 \cup (w_1, u_3) \cup P_3$ and $(w_2, w_3) \cup P_4$ are paths between w_2 and v_i and pairwise internally disjoint in $G_2' \cup \{e_3\}$. Therefore $\kappa_{G_2' \cup \{e_3\}}(w_2, v_i) \geq 3$.

We show $\kappa_{G_2'}(u_3, w_3) \geq 3$. By Lemma 6.1 and $\kappa(G_2 - e_3) \geq 2$, $G_2'[T_2 \cup S_i]$ has two internally disjoint paths P_1 and P_2 between u_3 and w_3. Note that $G_2'[T_1']$ with $T_1' \subset T_1$ has a path P_3 between w_1 and w_2. From construction, all of P_1, P_2, and $(u_3, w_1) \cup P_3 \cup (w_2, w_3)$ are paths between u_3 and w_3, and internally disjoint. Therefore $\kappa_{G_2'}(u_3, w_3) \geq 3$ holds.

(3) It is not difficult to observe that $p(G_2' - S_i) < p(G_2 - S_i)$ holds for the graph G_2' obtained from G_2 by both switchings $\{(v^*, v_i), (w_1, u_3), (w_2, w_3)\}/\{e_1, e_2, e_3\}$ and $\{(v^*, v_i), (w_1, w_3), (w_2, u_3)\}/\{e_1, e_2, e_3\}$, because $(G_2 - e_1 - e_2)[T_1]$ remains connected as observed above. □

Step III can be executed in $O(n^2m + n^3 \log n)$ time as follows. The number of switchings is $O(n)$, since each iteration decreases the number of S-components by at least one for some disconnecting pairs S. So it suffices to show that each switching of edges can be executed in $O(nm + n^2 \log n)$ time.

The ℓ-edge-connectivity in a given graph can be checked in $O(nm + n^2 \log n)$ time [19]. For two vertices $x, y \in V$, testing whether $\kappa_G(x, y) \geq 3$ can be done in $O(n^{1/2}m)$ time using a maximum flow technique [4]. Therefore, for a subset F' where $|F'|$ is a constant, we can test whether a shifting F''/F' is improving in a graph $G' = (V, E')$ or not in $O(nm + n^2 \log n)$ time. It is not difficult to see that we can find a pair of two edges satisfying the conditions in Properties 3.3 or 3.6.

We can easily find a pair of edges in $F_2 - F^*(G_2)$ satisfying the condition (b) of Property 3.3 as long as G_2 has two disjoint disconnecting pairs. If there is no disjoint disconnecting pair, then the total number of S-components over all disconnecting pairs is $O(n)$. For each S-component, we can identify all edges in $F_2 - F^*(G_2)$ s-admissible with respect to S in $O(m)$ time by applying a triconnected component algorithm [8]. From this observation, an improving switching which satisfies one of (a)-(e) can be found in $O(nm)$ time.

7 Justification for Step IV

Now Property 3.4 says that $F_3 - F^*(G_3)$ and a family $\mathcal{D}(G_3)$ of all minimal tight sets in G_3 satisfy

$$T \cap V[F_3 - F^*(G_3)] \neq \emptyset \text{ for all } T \in \mathcal{D}(G_3), \tag{7.1}$$

corresponding (3.2) and (3.3). To show that Step IV works correctly, we need to prove Properties 3.7 and 3.10.

Proof of Property 3.7: (1) Without loss of generality, assume that there is an edge $e = (v_1, u) \in F_3 - F^*(G_3)$ with $u \neq v^*$. Clearly, u is contained in an S_1-component T. Then G_3 has an edge $(v^*, w) \in F_3 - F^*(G_3)$ with $w \neq v_1$, by the definition of v^*. If w is in an S_1-component $T' \neq T$, then edges e and (v^*, w) satisfy condition (d) of Property 3.3, contradicting the assumption on G_3. Thus all edges $(v^*, w) \in F_3 - F^*(G_3)$ incident to v^* satisfy $w \in T$. Then G_3 has another S_1-component $T' \neq T$ that contains $e' \in (F_3 - F^*(G_3)) \cap E(G_3[T' \cup \{v_1\}])$, since S_1 is a disconnecting pair in G_3 satisfying (3.4). If e' is not incident to S_1, then these edges $e, (v^*, w)$ and e' satisfy condition (e) of Property 3.6, contradicting the assumption on G_3. If e' is incident to S_1, then edges e' and (v^*, w) satisfy condition (d) of Property 3.3, contradicting the assumption on G_3.

(2) Assume $F^a(S_i; G_3) \neq \emptyset$ holds for a disconnecting pair S_i in G_3 and G_3 has an S_i-component T_1 containing an edge $e_1 \in F^a(S_i; G_3)$. If there is another S_i-component $T_2 (\neq T_1)$

containing $e_2 \in F_3 - F^*(G_3)$ with $V[e_2] \subseteq T_2$, then two edges e_1 and e_2 satisfy condition (a) of Property 3.3, a contradiction. Hence by (7.1), all edges $e \in F_3 - F^*(G_3)$ that are contained in S_i-components other than T_1 are incident to S_i. Moreover, by (1), such edge e always is incident to v^*.

(3) Let $e = (u, w) \in F^*(G_3)$. Now $G_3 - e$ is not 2-vertex-connected and has a disconnecting vertex v. Assume that $v \neq v^*$. Note that $\{u, v\}$ or $\{w, v\}$ (say, $\{u, v\}$) is a disconnecting pair in G_3. Every disconnecting pair in G_3 has a common vertex v^* and hence $u = v^*$. Note that $\Gamma_{G_3}(w) = \{v, v^*\}$ holds, since otherwise $\{v, w\}$ is a disconnecting pair in G_3 that does not contain v^*, contradicting (3.5). By (7.1) and $(u, w) \in F^*(G_3)$, there is an edge $(v, w) \in F_3 - F^*(G_3)$, contradicting (2). Therefore $v = v^*$ holds. Since $\{u, v^*\}$ is a disconnecting pair in G_3 or $\Gamma_{G_3}(u) = \{w, v^*\}$, the property is proved. □

Proof of Property 3.9: By $\lambda(G_3) \geq \ell$, we have $c_{G_3}(v^*) \geq \ell$. Assume that $c_{G_3}(v^*) = \ell$ holds. Property 3.7 (3) says that if G_3 has an edge $e = (u, w) \in F^*(G_3)$, then there is a cut $T \subset V$ with $u \in T$ and $\Gamma_{G_3}(T) = \{v^*, w\}$ that is a disconnecting pair in G_3. By $\lambda(G_3) \geq \ell$ and $c_{G_3}(u, w) = 1$, we have $c_{G_3}(T) \geq \ell$ and $c_{G_3}(T, v^*) \geq \ell - 1$. Let $T' = V - T - \{v^*, w\}$. By $c_{G_3}(v^*) = \ell$ and $T' \neq \emptyset$, we have $c_{G_3}(T', v^*) = 1$. Let $\Gamma_{G_3}(v^*) \cap T' = \{t'\}$. Now $T' = \{t'\}$ holds, since otherwise $\Gamma_{G_3}(T' - t') = \{t', w\}$ that is a disconnecting pair in G_3 and does not contain v^*, contradicting (3.5). Property 3.7 (3) says that (v^*, t') is not an edge in $F^*(G_3)$ and hence $(v^*, t') \in E$. By (7.1), $E_{G_3}(w, t')$ contains an edge in $F_3 - F^*(G_3)$, contradicting Property 3.7 (1). □

Proof of Property 3.10: Assume that G_3 satisfies none of conditions in Properties 3.3, 3.6 or 3.8. If G_3 has an edge in F_3 incident to v^*, Property 3.9 implies $c_{G_3}(v^*) > \ell$ and hence condition of Property 3.8 holds, a contradiction.

We consider the case where no edge in F_3 is incident to v^*. We first show that G_3 has at least two disconnecting pair. Otherwise G_3 has exactly one disconnecting pair $S = \{v^*, v_1\}$ and, for each edge $e = (u, w) \in F^*(G_3)$, $u = v_1$ without loss of generality and w is a minimal tight set with $\Gamma_{G_3}(w) = \{v^*, u\}$ in G_3 from Property 3.7 (3). By (7.1) and $(u, w) \in F^*(G_3)$, there is an edge $(v^*, w) \in F_3 - F^*(G_3)$, contradicting assumption that no edge in F_3 is incident to v^*.

Now G_3 has two disconnecting pairs S_1 and S_2 and hence there are an S_1-component T_1 and an S_2-component T_2 with $T_1 \cap T_2 \neq \emptyset$. By (7.1), G_3 has two edges in $e_1, e_2 \in F_3 - F^*(G_3)$ such that $V(e_i) \cap T_i \neq \emptyset$ holds and e_i is not incident to v^* for $i = 1, 2$ since no edge in F_3 is incident to v^*. These two edges e_1 and e_2 satisfy condition (c) in Property 3.3, a contradiction. □

We show Property 3.12. For this, we prove the following two claims.

Claim 7.1 *Assume that G_4 has two distinct S_i-components T_1 and T_2 in G_4 such that there are two edges $e_1 \in E(G_4[T_1]) \cap (F^n(S_i; G_4) - E_{G_4}[S_i])$ dividing with respect to S_i and $e_2 \in E_{G_4}(v^*, T_2) \cap (F_4 - F^*(G_4))$. Then by switching two edges e_1 and e_2, the resulting graph G'_4 is ℓ-edge-connected and every pair S' of two vertices in V satisfies $p(G_4 - S') \geq p(G'_4 - S')$.* □

Proof: First we see $\kappa(G_4 - \{e_1, e_2\}) \geq 2$ from Property 3.5. Let X_1 and X_2 be two S_i-components in $G_4 - e_1$ with $T_1 = X_1 \cup X_2$ and $E_{G_4}(X_1, X_2) = \{e_1\}$ (note $E(G_4[X_1]) \cap (F^n(S_i; G_4) - E_{G_4}[S_i]) \neq \emptyset \neq E(G_4[X_2]) \cap (F^n(S_i; G_4) - E_{G_4}[S_i])$ since $e_1 \in F^n(S_i; G_4) - E_{G_4}[S_i]$ is dividing with respect to S_i). Let $S_i = \{v^*, v_i\}$, $e_1 = (u_1, u_2)$, $e_2 = (v^*, w_2) \in E_{G_4}(v^*, T_2) \cap (F_4 - F^*(G_4))$, $f_j = (x_j, y_j) \in E(G_4[X_j]) \cap (F^n(S_i; G_4) - E_{G_4}[S_i])$ and X_{j1} and X_{j2} be S_i-components in $G_4 - \{e_1, f_j\}$ such that we have $X_j = X_{j1} \cup X_{j2}$, $E_{G_4}(X_{j1}, X_{j2}) = \{f_j\}$, $\{u_j, x_j\} \subseteq X_{j1}$ and $y_j \in X_{j2}$ for $j = 1, 2$. From $f_j \notin F^*(G_4)$, we see

$$c_{G_4}(v^*, X_{j2}) > 0 \text{ and } c_{G_4}(v_i, X_{j2}) > 0 \text{ for } j = 1, 2. \tag{7.2}$$

By applying Claim 6.2 to G_4, we can switch two edges e_1 and e_2 while preserving the ℓ-edge-connectivity of G_4. Let $G'_4 := G_4 - \{e_1, e_2\} + \{(v^*, u_1), (u_2, w_2)\}$ satisfy $\lambda(G'_4) \geq \ell$ without loss of generality. Let $\mathcal{S}(H)$ be a family of sets S of two vertices such that S disconnects some two vertices $x, y \in V$ with $\kappa_{G_4}(x, y) \geq 3$ and $\kappa_H(x, y) = 2$ in H. We claim that $\mathcal{S}(G'_4) = \{S_i\}$ and $p(G_4 - S_i) = p(G'_4 - S_i)$, implying Property 7.1. For this, after proving (I) $\mathcal{S}(G'_4 + e_1) = \emptyset$, we show (II) $\mathcal{S}(G'_4) = \{S_i\}$. Now by $\kappa(G_4 - \{e_1, e_2\}) \geq 2$, G_4 have the following paths P_1 – P_6. $(G_4 - \{e_1, e_2\})[T_2 \cup S_i]$ has two internally disjoint paths P_1 and P_2 such that P_1 (resp., P_2) connects w_2 and v^* (resp., v_i). By (7.2), $(G_4 - \{e_1, e_2\})[X_1 \cup S_i]$ has a path P_3 which connects v_i and v^*. $(G_4 - \{e_1, e_2\})[X_2 \cup S_i]$ has two internally disjoint paths P_4 and P_5 such that P_4 (resp., P_5) connects between u_2 and v^* (resp., v_i).

(I) We first show $\mathcal{S}(G'_4 + e_1) = \emptyset$. Clearly, $\mathcal{S}(G'_4 + \{e_1, e_2\}) = \emptyset$ holds. So it suffices to show $\kappa_{G'_4 + e_1}(v^*, w_2) \geq 3$. We see this property since $G'_4 + e_1$ has three internally disjoint paths P_1, $P_2 \cup P_3$ and $(w_2, u_2) \cup P_4$ between w_2 and v^*.

(II) We next show $\mathcal{S}(G'_4) = \{S_i\}$. Note that every $S \in \mathcal{S}(G'_4)$ disconnects u_1 and u_2 in G'_4 by $\mathcal{S}(G'_4 + e_1) = \emptyset$. Clearly, $S_i \in \mathcal{S}(G'_4)$ holds. If $\mathcal{S}(G'_4) - \{S_i\} \neq \emptyset$ holds, then $S \in \mathcal{S}(G'_4) - \{S_i\}$ disconnects two vertices (i) u_2 and v^*, (ii) u_2 and v_i, (iii) u_1 and v^* or (iv) u_1 and v_i. Hence we show (i) $\kappa_{G'_4}(u_2, v^*) \geq 3$, (ii) $\kappa_{G'_4}(u_2, v_i) \geq 3$, (iii) $\kappa_{G'_4}(u_1, v^*) \geq 3$ and (iv) $\kappa_{G'_4}(u_1, v_i) \geq 3$. We first show (i) and (ii). Three paths P_4 , $P_5 \cup P_3$ and $(u_2, w_2) \cup P_1$ proves (i) and three paths P_5 , $P_4 \cup P_3$ and $(u_2, w_2) \cup P_2$ proves (ii). We see that (iii) holds by $\kappa(G_4 - \{e_1, e_2\}) \geq 2$ and $(u_1, v^*) \in E(G'_4)$. We show (iv). We claim that $\kappa_{G'_4[X_1 \cup \{v_i\}]}(u_1, v_i) \geq 2$ holds. Otherwise $G'_4[X_1 \cup \{v_i\}]$ has a cut vertex $z \in X_1$ which disconnects u_1 and v_i. Since we have (7.2) and $G'_4[X_1]$ is connected, we have $c_{G'_4}(X_{11}, v_i) = 0$ or $G'_4[X_{11}]$ has a cut $Y \subset X_{11}$ with $\Gamma_{G'_4[X_{11}]}(Y) = \{z\}$, $u_1 \in Y$ and $x_1 \notin Y$. In the former case, we have $E_{G_4}(X_{11}) = E_{G_4}(X_{11}, v^*) \cup \{e_1, f_1\}$ and hence $c_{G_4}(X_{11}, v^*) \geq \ell - 2$ by $c_{G_4}(X_{11}) \geq \ell$. Now we have $c_{G_4}(v^*, V - X_{11}) \geq 3$ by (7.2) and $c_{G_4}(v^*, T_2) > 0$. This contradicts $c_{G_4}(v^*) = \ell$. In the latter case, we have $\Gamma_{G_4}(Y) = \{u_2, z\}$, which contradicts (3.5). Hence we have $\kappa_{G'_4[X_1 \cup \{v_i\}]}(u_1, v_i) \geq 2$. Now we have another path $(u_1, v^*) \cup P_8$ which connects u_1 and v_i. This implies $\kappa_{G'_4}(u_1, v_i) \geq 3$.

Finally, it is not difficult to see $p(G'_4 - S_i) = p(G_4 - S_i)$, since $(G_4 - e_2)[T_2]$ is connected by $e_2 \in F_4 - F^*(G_4)$ and G'_4 has an edge $(u_2, w_2) \in E_{G'_4}(X_2, T_2)$. □

Claim 7.2 *Assume that for some disconnecting pair S_i, G_4 has two distinct S_i-components T_1 and T_2 in G_4 such that there are an edge $e_1 = (u_1, u_2) \in E(G_4[T_1]) \cap (F^n(S_i; G_4) - E_{G_4}[S_i])$ and an edge $e_2 = (v^*, w_2) \in E_{G_4}(v^*, T_2) \cap (F_4 - F^*(G_4))$. Let X_1 and X_2 be two S_i-components* *in* $G_4 - e_1$ *with* $T_1 = X_1 \cup X_2$, $E_{G_4}(X_1, X_2) = \{e_1\}$, $u_1 \in X_1$ *and* $u_2 \in X_2$. *Let* $G'_4 := G_4 - \{e_1, e_2\} + \{(v^*, u_1), (u_2, w_2)\}$ *satisfy* $\lambda(G'_4) \geq \ell$ *without loss of generality* (*such* G'_4 *exists by Claim 6.2*). *Then every pair* S *of two vertices with* $p(G'_4 - S) > p(G_4 - S)$ *satisfies* $S = \{v^*, x\}$ *with* $x \in X_1$ *and* $p(G'_4 - S) = p(G_4 - S) + 1$ *and disconnects* u_1 *and* v_i *in* $G_4[X_1 \cup S_i]$. □

Proof: By $\kappa(G_4 - e_1) \geq 2$, we have $c_{G_4}(v^*, X_j) > 0$ and $c_{G_4}(v_i, X_j) > 0$ for $j = 1, 2$. By using this property instead of (7.2), we see $\kappa_{G'_4+e_1}(w_2, v^*) \geq 3$, $\kappa_{G'_4}(u_2, v^*) \geq 3$, $\kappa_{G'_4}(u_2, v_i) \geq 3$ and $\kappa_{G'_4}(u_1, v^*) \geq 3$, in a similar way to the proof of Claim 7.1. Note that the condition that $e_1 \in (F^n(S_i; G_4) - E_{G_4}[S_i])$ is dividing with respect to S_i is used only for proving $\kappa_{G'_4}(u_1, v_i) \geq 3$ in the proof of Claim 7.1. In this case, $\kappa_{G'_4}(u_1, v_i) = 2$ possibly holds. Then from $\kappa_{G'_4}(u_2, v^*) \geq 3$, $\kappa_{G'_4}(u_2, v_i) \geq 3$ and $\kappa_{G'_4}(u_1, v^*) \geq 3$, it is not difficult to see that every pair S of two vertices with $p(G'_4 - S) > p(G_4 - S)$ satisfies $S = \{v^*, x\}$ with $x \in X_1$ and $p(G'_4 - S) = p(G_4 - S) + 1$ and disconnects u_1 and v_i in $G_4[X_1 \cup S_i]$. □

Proof of Property 3.12: There are the following three possible cases (1) – (3).

(1) $F^a(S_1; G_4) \neq \emptyset$. Let T_1 be an S_1-component in G_4 with $E(G[T_1]) \cap F^a(S_1; G_4) \neq \emptyset$. Note that since none of conditions in Property 3.3 holds in G_4, we have $F^a(S_1; G_4) \cup (F^n(S_1; G_4) - E_{G_4}[S_1]) \subseteq E(G_4[T_1])$. Hence every edge $e \in F_4 - F^*(G_4)$ with $V[e] - T_1 - S_1 \neq \emptyset$ satisfies $v^* \in V[e]$. Let $e_1 \in F^n(S_1; G_4) - E_{G_4}[S_1]$ dividing with respect to S_1 and $e_2 \in E_{G_4}(v^*, T_2) \cap (F_4 - F^*(G_4))$ with some $T_2 \in \mathcal{C}(G_4 - S_1) - T_1$. By applying Claim 7.1, we can obtain G'_4 by switching e_1 and e_2 so that $\lambda(G'_4) \geq \ell$ holds and $p(G'_4 - S) \leq p(G_4 - S)$ holds for each pair S of two vertices in V. Moreover, since e_1 is dividing, G'_4 has two S_1-components X_1 and $X'_2 = X_2 \cup T_2$ without loss of generality, where X_1 and X_2 denote S-components in $G_4 - e_1$ with $T_1 = X_1 \cup X_2$ such that $E(G_4[X_j]) \cap (F^n(S_1; G_4) - E_{G_4}[S_1]) \neq \emptyset$ holds for $j = 1, 2$. Now we have $E(G[T_1]) \cap F^a(S_1; G_4) \neq \emptyset$, which implies that we can apply Property 3.3 (a) to G'_4. The graph G''_4 resulting from a switching of Property 3.3 (a) in G'_4 satisfies $p(G''_4 - S) \leq p(G_4 - S)$ for every pair of two vertices in V and $p(G''_4 - S_1) = p(G'_4 - S_1) - 1 = p(G_4 - S_1) - 1$.

(2) $F^a(S_1; G_4) = \emptyset$ holds and G_4 has another disconnecting pair $S_2 = \{v^*, v_2\} \neq S_1$ such that $G_4[T_1]$ contains an edge $e_1 \in F^n(S_1; G_4) - E_{G_4}[S_1]$ dividing with respect to S_1 and $v_2 \in T_2$ hold for some two distinct S_1-components T_1 and T_2. Note that G_4 has an S_2-component $T' \subset T_2$. Let $e_2 \in E_{G_4}(v^*, T') \cap (F_4 - F^*(G_4))$ (note that such e_2 exists from (7.1) and $v^* \in V[e_2]$ holds since otherwise the condition (c) in Property 3.3 would hold in G_4). By applying Claim 7.1, we can obtain G'_4 by switching e_1 and e_2 such that $\lambda(G'_4) \geq \ell$ holds and $p(G'_4 - S) \leq p(G_4 - S)$ holds for each pair of two vertices in V. Moreover, it is not difficult to see $p(G'_4 - S_2) = p(G_4 - S_2) - 1$, since e_1 is s-admissible with respect to S_2 in G'_4.

(3) $F^a(S_1; G_4) = \emptyset$ holds, G_4 has a disconnecting pair S_i other than S_1, some S_1-component T_1 in G_4 contains an edge $F^n(S_1; G_4) - E_{G_4}[S_1]$ dividing with respect to S_1 and every $S_i = \{v^*, v_i\} (\neq S_1)$ satisfies $v_i \in T_1$. We first consider the case that some S_1-component $T_2 \in \mathcal{C}(G_4 - S_1) - \{T_1\}$ has has an edge $e_1 = (u_1, u_2) \in F^n(S_1; G_4) \cap E(G_4[T_2]) \neq \emptyset$. Now G_4 has an S_2-component $T' \subset T_1$ with $S_2 \neq S_1$ and an edge $e_2 = (v^*, w_2) \in E_{G_4}(v^*, T') \cap (F_4 - F^*(G_4))$ (note that such e_2 exists from (7.1) and $v^* \in V[e_2]$ holds since otherwise the condition (c) in Property 3.3 would hold in G_4). By applying Claim 6.2, we can obtain a graph G'_4 such

that $\lambda(G'_4) \geq \ell$ by switching e_1 and e_2; $G_4 - \{e_1, e_2\} + \{(u_1, v^*), (u_2, w_2)\}$ is assumed without loss of generality. Let X_1 and X_2 be two S_1-components in $G_4 - e_1$ with $T_2 = X_1 \cup X_2$, $E_{G_4}(X_1, X_2) = \{e_1\}$, $u_1 \in X_1$ and $u_2 \in X_2$. Claim 7.2 says that every $S \in \mathcal{S}(G'_4)$ satisfies $p(G'_4 - S) = p(G_4 - S) + 1$ and $S = \{v^*, x\}$ with $x \in X_1$ and disconnects u_1 and v_i in $G_4[X_1 \cup S_1]$, where $\mathcal{S}(G'_4)$ denotes a family of sets defined in the proof of Claim 7.1. Note that we have $p(G'_4 - S_2) = p(G_4 - S_2) - 1$ (by $e_1 \in F^a(S_2; G_4)$) and $T_1 \cup X_2$ and X_1 are both S_1-components in G'_4. Let $e_3 \in F^n(S_1; G_4) \cap E(G_4[T_1])$ be an edge dividing with respect to S_1. By applying Claim 7.1 to G'_4, we obtain a graph G''_4 by switching e_3 and (v^*, u_1) such that $\lambda(G''_4) \geq \ell$ holds and every pair of two vertices in V satisfies $p(G''_4 - S) \leq p(G'_4 - S)$. Moreover, it is not difficult to see that $p(G''_4 - S') = p(G'_4 - S') - 1 = p(G_4 - S)$ for every $S' \in \mathcal{S}(G'_4)$, since e_3 is s-admissible with respect to S' in G'_4. Hence we have $p(G''_4 - S_2) \leq p(G'_4 - S_2) = p(G_4 - S_2) - 1$, which proves this property.

We finally consider the case that every S_1-component $T (\neq T_1)$ has no edge $e \in F^n(S_1; G_4) - E_{G_4}[S_1]$. Let $e_1 \in F^n(S_1; G_4) \cap E(G_4[T_1])$ be an edge dividing with respect to S_1. and X_1 and X_2 be two S_1-components in $G_4 - e_1$ with $T_1 = X_1 \cup X_2$, $E_{G_4}(X_1, X_2) = \{e_1\}$, $u_1 \in X_1$ and $u_2 \in X_2$. Let $T_2 \in \mathcal{C}(G_4 - S_1) - \{T_1\}$. Then by (7.1), there is an edge $e_2 = (v^*, w_2) \in E_{G_4}(v^*, T_2) \cap (F_4 - F^*(G_4)) \neq \emptyset$. By applying Claim 7.1 to G_4, we obtain a graph $G'_4 = G_4 - \{e_1, e_2\} + \{(v^*, u_1), (u_2, w_2)\}$ without loss of generality by switching e_1 and e_2 such that $\lambda(G'_4) \geq \ell$ holds and every pair of two vertices in V satisfies $p(G'_4 - S) \leq p(G_4 - S)$. If G'_4 has no edge in $F^n(S_1; G'_4) - E_{G'_4}[S_1]$ dividing with respect to S_1, then this proves the property. Otherwise X_1 and $X_2 \cup T_2$ are both S_1-components containing an edge in $F^n(S_1; G'_4) - E_{G'_4}[S_1]$ since $e_1 \in F^n(S_1; G_4)$ is dividing. Hence we see that G'_4 satisfies (2) or the first case of (3) and we can obtain the desired graph G''_4 by applying the switching operations in (2) or the first case of (3). □

We can show that Step IV can be executed in $O(n^2 m + n^3 \log n)$ time, by a similar observation in Section 6, since the complexity depends on shiftings of edges.

8 Justification for Step V

Let $G_5 = (V, E \cup F_5)$ be obtained from G_4 by Step IV. The graph G_5 is ℓ-edge-connected and has disconnecting pairs $S_i = \{v^*, v_i\}$ $(i = 1, \ldots, q)$, where the number of edges in F_5 incident to v^* is $\max\{0, \ell - c_G(v^*)\}$. Moreover, G_5 satisfies $F^*(G_5) = \emptyset$ and one of (i) and (ii) in Property 3.13. We will prove Properties 3.14 and 3.15, from which the correctness of Step V follows.

Proof of Property 3.14: Let $F'_5 \subseteq F_5$ be the set of edges incident to S_1, and let $F''_5 = F_5 - F'_5$. By Property 3.13, the number of edges in F_5 incident to v^* is $\max\{\ell - c_G(v^*), 0\}$. By Property 3.13 and choice of v^* in the case of $q = 1$, any edge in F_5 incident to v_1 is also incident to v^*. Hence the number of edges in F_5 incident to S_1 is $\max\{\ell - c_G(v^*), \ell - c_G(v_1), 0\}$. Hence it is sufficient to show $p(G - S_1) = p(G_5 - S_1) + |F''_5|$. If $p(G - S_1) < p(G_5 - S_1) + |F''_5|$ holds, then there is at least one edge $e \in F''_5$ such that $p((G_5 - e) - S_1) = p(G_5 - S_1)$. Then this e is s-admissible with respect to S_1 since e is not incident to S_1. This contradicts Property 3.13(i) of G_5. $\square$

Proof of Property 3.15: Assume that G_5 satisfies (ii) of Property 3.13. We first show that

$$|\mathcal{D}(G_5)| \leq \ell/2. \tag{8.1}$$

Note that every $D \in \mathcal{D}(G_5)$ satisfies $c_{G_5}(v^*, D) > 0$ by (3.5). We claim that $c_{G_5}(v^*, D) \geq 2$ holds for every $D \in \mathcal{D}(G_5)$. If $E_{G_5}(v^*, D) \cap F_5 \neq \emptyset$ holds, then we have $c_{G_5}(v^*, D) \geq 2$ since $c_{G_5}(v^*, D) = 1$ implies that the edge e with $\{e\} = E_{G_5}(v^*, D) \subseteq F_5$ satisfies $e \in F^*(G_5)$, a contradiction. If $E_{G_5}(v^*, D) \cap F_5 = \emptyset$ and $c_{G_5}(v^*, D) = 1$ hold, then we have $|D| = 1$, since otherwise $D - u$ contradicts the minimality of D for $\{u\} = D \cap \Gamma_{G_5}(v^*)$. Let $D = \{u\}$. By (3.4) and $(u, v^*) \in E$, we have $(u, v_i) \in F_5 - F^*(G_5)$, where $\Gamma_{G_5}(D) = \{v^*, v_i\}$. This contradicts Property 3.7 (1). Thus, $c_{G_5}(v^*, D) \geq 2$ holds for every $D \in \mathcal{D}(G_5)$. From $c_{G_5}(v^*) = \ell$, we have (8.1).

We now prove the following inequalities.

$$|F_5| + |\mathcal{D}(G_5)| - 1 \leq 5\ell/2 - 1, \tag{8.2}$$

$$|F_5| \leq 2\ell. \tag{8.3}$$

We first consider the case of $F^a(S_i; G_5) \neq \emptyset$ for each disconnecting pair S_i in G_5. Let T_1 be an S-component in G_5 containing an edge in $F^a(S; G_5)$, such that no S'-component $T' \subset T_1$ with $S' \neq S$ contains an edge in $F^a(S'; G_5)$, where S and S' are disconnecting pairs in G_5. By Property 3.11, we see $|F^a(S; G_5)| \leq c_G(v^*, T_1)$. Now we claim that $|F^n(S; G_5) - E_{G_5}[S]| \leq c_{G_5}(v^*,\ T_1)$ holds. The removal of an edge in $F^n(S; G_5) - E_{G_5}[S]$ increases the number of S-components. Moreover, since none of edges in $F^n(S; G_5) - E_{G_5}[S]$ is dividing with respect to S, for each $e \in F^n(S; G_5) - E_{G_5}[S]$, at least one of $G_5[X_1]$ and $G_5[X_2]$ contains no edge in $F^n(S; G_5) - E_{G_5}[S]$ and we have $c_{G_5}(v^*, X_i) > 0$ for $i = 1, 2$, where X_1 and X_2 are S-components in $G_5 - e$ with $E_{G_5}(X_1, X_2) = \{e\}$. This implies $|F^n(S; G_5) - E_{G_5}[S]| \leq c_{G_5}(v^*,\ T_1)$. Note that $c_G(v^*, T_1) + c_G(v^*, V - T_1) + |F^n(S; G_5) \cap E_{G_5}[S]| = \ell$ holds.

From the above relations, we have $|F_5| = |F^a(S; G_5)| + |F^n(S; G_5) - E_{G_5}[S]| + |F^n(S; G_5) \cap E_{G_5}[S]| \leq c_G(v^*, T_1) + c_{G_5}(v^*, T_1) + |F^n(S; G_5) \cap E_{G_5}[S]| = c_{G_5}(v^*, T_1) + \ell - c_G(v^*, V - T_1)$. Now we have $c_{G_5}(v^*, V - T_1) \geq 2$, since every $D \in \mathcal{D}(G_5)$ satisfies $c_{G_5}(v^*, D) \geq 2$. From this and $c_{G_5}(v^*) = \ell$, it follows $c_{G_5}(v^*, T_1) - c_G(v^*, V - T_1) \leq \ell - 2$. So $|F_5| \leq 2\ell - 2$ holds, proving (8.3). By (8.1), we have $|F_5| + |\mathcal{D}(G_5)| - 1 \leq 5\ell/2 - 3$, proving (8.2).

Finally, we consider the case that $F^a(S; G_5) = \emptyset$ holds for some disconnecting pair S. In a similar way to the above, we see $|F^n(S; G_5) - E_{G_5}[S]| \leq c_{G_5}(v^*)$ since none of edges in $F^n(S; G_5) - E_{G_5}[S]$ is dividing with respect to S. Hence we have $|F_5| = |F^n(S; G_5) - E_{G_5}[S]| + |F^n(S; G_5) \cap E_{G_5}[S]| \leq c_{G_5}(v^*) + |F^n(S; G_5) \cap E_{G_5}[S]| \leq 2\ell$, proving (8.3). We have $|F_5| + |\mathcal{D}(G_5)| - 1 \leq 5\ell/2 - 1$ by (8.1), proving (8.2). $\square$

In Step V, if G_5 satisfies (i) of Property 3.13, then the edge set F_6 can be found in linear time. Otherwise we inspect all possible choices of F with $|F| \leq 5\ell/2 - 1$. The complexity of Step V is $O((5\ell/2-1)n^{2(5\ell/2-1)+3})$ time.

9 Concluding Remarks

In this paper, by using the algorithms EV-AUGMENT [14] and EV-AUGMENT3 [15] as ingredients, we gave an algorithm for augmenting a given arbitrary graph G to an ℓ-edge-connected and 3-vertex-connected graph by adding the smallest number of new edges. However, our lower bound on the optimal value does not always give the exact optimal value. So, it is desired to find a new and stronger lower bound on the optimal value.

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A Proof of Claim 4.2

To prove Claim 4.2, we first observe some properties about a graph G' satisfying (3.2) and (3.3). Let $G' = (V \cup \{s\}, E \cup F)$ be a graph with $G = G' - s$ and $F = E_{G'}(s)$ which satisfies (3.2) and (3.3). An edge $e \in F$ is called *κ-critical* if $G' - e$ violates (3.2) or (3.3).

Property A.1 *For each κ-critical edge $e = (s,t) \in E_{G'}(s)$, G' has a cut $T \subset V$ with $V - T - \Gamma_G(T) \neq \emptyset$ such that, for a neighbor $t \in T$ of s,*

(I) $|\Gamma_G(T)| = 2$ *and* $E_{G'}(s,T) = \{(s,t)\}$,
(II) $|\Gamma_G(t)| = 1$, $c_{G'}(s,t) = 2$ *and* $T = \{t\}$ *or*
(III) $|\Gamma_G(t)| = 0$, $c_{G'}(s,t) = 3$ *and* $T = \{t\}$. □

Proof: If $G' - e$ violates (3.3) for $e = (s,t) \in E_{G'}(s)$, then the vertex t clearly satisfies (II) or (III). If $G' - e$ violates (3.2) for $e = (s,t) \in E_{G'}(s)$, then there is a cut $T \subset V$ with $V - T - \Gamma_G(T) \neq \emptyset$ such that either (a) $|\Gamma_G(T)| = 2$ and $E_{G'}(s,T) = \{(s,t)\}$ (which satisfies (I)), (b) $|\Gamma_G(T)| = 1$, $E_{G'}(s) \cap T = \{t,u\}$ $(t \neq u)$ and $c_{G'_1}(s,t) = 1$ or (c) $|\Gamma_G(T)| = 0$, $E_{G'}(s) \cap T = \{t,u,v\}$ $(t \neq u \neq v \neq t)$ and $c_{G'}(s,t) = 1$. In the case of (b) (resp., (c)), the cut $T - u$ (resp., $T - \{u,v\}$) satisfies (I). □

A cut $T \subseteq V$ satisfying (I), (II) or (III) in Property A.1 is called *κ-critical.* Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$, $\mathcal{T}_3$) denote the family of all κ-critical cuts T of type (I) (resp., (II), (III)) such that no $T' \subset T$ is of type (I) (resp., (II), (III)). We observe the following properties about $\mathcal{T}_1$.

Property A.2 *If $T_1, T_2 \in \mathcal{T}_1$ intersect each other, then $T_3 = T_1 \cup T_2$ satisfies $V - T_3 - \Gamma_G(T_3) \neq \emptyset$, $|\Gamma_G(T_3)| = 1$ and $c_{G'}(s,T_3) = |\Gamma_{G'}(s) \cap T_3| = 2$.*

Proof: We first show $|\Gamma_G(T_1 \cap T_2)| \geq 3$. Since we have $c_{G'}(s, T_1 \cap T_2) \leq c_{G'}(s,T_1) = 1$ and G' satisfies (3.2), we have $|\Gamma_G(T_1 \cap T_2)| \geq 2$. Moreover, if $|\Gamma_G(T_1 \cap T_2)| = 2$ holds, then we see $c_{G'}(s, T_1 \cap T_2) = 1$, contradicting the minimality of T_1. Hence by (2.2), $T_3 = T_1 \cup T_2$ satisfies $|\Gamma_G(T_3)| \leq 1$.

We claim $V - T_3 - \Gamma_G(T_3) \neq \emptyset$. Assume that $V - T_3 - \Gamma_G(T_3) = \emptyset$ holds. If $\Gamma_G(T_3) = \emptyset$ holds, then it is not difficult to see that $V - T_1 - \Gamma_G(T_1) (\neq \emptyset)$ contradicts the minimality of T_2. If $\Gamma_G(T_3) \neq \emptyset$ holds, then $\Gamma_G(T_3) = \{v\} \subseteq \Gamma_G(T_1)$ holds without loss of generality (note $|\Gamma_G(T_3)| \leq 1$). Then $T' = V - T_1 - \Gamma_G(T_1) \neq \emptyset$ satisfies $T' \subset T_2$, $|\Gamma_G(T')| \leq 2$ and $c_{G'}(s,T') \leq c_{G'}(s,T_2) = 1$. By (3.2), we see $|\Gamma_G(T')| = 2$ and $c_{G'}(s,T') = 1$, contradicting the minimality of T_2.

We see $|\Gamma_G(T_3)| = 1$, since $|\Gamma_G(T_3)| = 0$ implies $c_{G'}(T_3) = c_{G'}(s,T_3) = 2$, contradicting (3.2) or (3.3). Moreover, T_3 satisfies $c_{G'}(s,T_3) = |\Gamma_{G'}(s) \cap T_3| = 2$ since G' satisfies $c_{G'}(s,T_3) \leq 2$ and (3.2). □

Property A.3 *Let $\mathcal{T}_4$ be a family of cuts $T \subseteq V$ such that $V - T - \Gamma_G(T) \neq \emptyset$, $|T| \geq 2$, $|\Gamma_G(T)| = 1$ and $c_{G'}(s,T) = |\Gamma_{G'}(s) \cap T| = 2$ hold and no $T' \subset T$ satisfies the property. If $T_1, T_2 \in \mathcal{T}_4$ intersect each other, then $T_3 = T_1 \cup T_2$ satisfies $V - T_3 - \Gamma_G(T_3) \neq \emptyset$, $|\Gamma_G(T_3)| = 0$ and $c_{G'}(s,T_3) = |\Gamma_{G'}(s) \cap T_3| = 3$.*

Proof: We first show that $G[T]$ is connected for every $T \in \mathcal{T}_4$. If $G[T]$ is not connected, then there is a partition $\{T', T''\}$ of T with $c_G(T', T'') = 0$. By $|\Gamma_G(T')| \le |\Gamma_G(T)| = 1$, $|\Gamma_G(T'')| \le |\Gamma_G(T)| = 1$, (3.2) and (3.3) in G', we have $c_{G'}(s, T') \ge 2$ and $c_{G'}(s, T'') \ge 2$ which contradicts $c_{G'}(s, T) = 2$.

Assume that two cuts $T_1, T_2 \in \mathcal{T}_4$ intersect each other in G. Let $\{v_i\} = \Gamma_G(T_i)$ for $i = 1, 2$. Since $G[T_i]$ is connected for $i = 1, 2$, we have $v_1 \in T_2$ and $v_2 \in T_1$. Hence $C = T_1 \cup T_2$ is a connected component in G. Then it is not difficult to see $T_2 - T_1 - v_1 = \emptyset = T_1 - T_2 - v_2$ from the minimality of T_1 and T_2. Thus we have $C = T_1 \cup \{v_1\} = T_2 \cup \{v_2\}$, from which $V - C = V - T_1 - v_1 \neq \emptyset$. Now we have $c_{G'}(s, T_1 \cap T_2) \ge 1$ by $|\Gamma_G(T_1 \cap T_2)| = 2$ and (3.2). Hence $c_{G'}(C) = c_{G'}(s, C) \le 3$ holds. We see that $T_3 = T_1 \cup T_2$ satisfies $c_{G'}(s, T_3) = |\Gamma_{G'}(s) \cap T_3|$ $= 3$ since (3.2) and $V - C \neq \emptyset$ imply $c_{G'}(C) \ge 3$. □

Let $\mathcal{T}_5$ be a family of cuts $T \subseteq V$ such that $V - T - \Gamma_G(T) \neq \emptyset$, $|T| \ge 3$, $|\Gamma_G(T)| = 0$ and $c_{G'}(s, T) = |\Gamma_{G'}(s) \cap T| = 3$ hold and no $T' \subset T$ satisfies the property. Note that every cut $T \in \mathcal{T}_i$ satisfies the condition $c_{G'}(s, T) = 3 - |\Gamma_G(T)|$ for $i = 1, \ldots, 5$.

Proof of Claim 4.2: Let $F'' = E_{G'}(s, X)$. Clearly, $c_{G'}(s, X) = |F''| \ge p(X)$ holds, since otherwise (4.1) violates. Hence it suffices to show $|F''| \le p(X)$.

From the minimality of F'', there is a κ-critical cut $T_v \subseteq X$ of type (I), (II) or (III) with $v \in T_v$ for each $(s, v) \in F''$. Let $\mathcal{T}_i$ be a family of cuts $T \subseteq X$ defined above for $i = 1, \ldots, 5$. Note that every edge $e \in F''$ satisfies $V[e] \cap V \subseteq T$ for some $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Then we obtain a family $\mathcal{T}'$ of cuts from $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ by replacing every pair of two cuts $T_1, T_2 \in \mathcal{T}_1$ satisfying $T_1 \cap T_2 \neq \emptyset$ with $T_3 \in \mathcal{T}_4$ satisfying $T_3 \subseteq T_1 \cup T_2$ (this is possible by Property A.2). Note that every two cuts in $\mathcal{T}' \cap \mathcal{T}_1$ are pairwise disjoint. This implies that every two cuts $T'_1 \in \mathcal{T}' \cap \mathcal{T}_1$ and $T'_2 \in \mathcal{T}' \cap \mathcal{T}_4$ satisfies $T'_1 \cap T'_2 = \emptyset$, since otherwise there is a cut $T'_3 \in \mathcal{T}_1$ with $T'_3 \subset T'_2$ and $T'_1 \cap T'_3 \neq \emptyset$, contradicting the construction of $\mathcal{T}'$. Next we replace every pair of two cuts $T_1, T_2 \in \mathcal{T}' \cap \mathcal{T}_4$ satisfying $T_1 \cap T_2 \neq \emptyset$ with $T_3 \in \mathcal{T}_5$ satisfying $T_3 = T_1 \cup T_2$ (this is possible by Property A.3). Then we observe that the resulting family $\mathcal{T}''$ is a subpartition of X which satisfies $V[F''] \cap X \subseteq \cup_{T \in \mathcal{T}''} T$. Hence we have $|F''| = \sum_{T \in \mathcal{T}''} (3 - |\Gamma_G(T)|) \le p(X)$ from the maximality of $p(X)$. □

B Algorithm for Finding F_1 in Step I

We present a graph theoretical algorithm finding a minimum set F_1 in Step I of algorithm EV-AUGMENT3*.

To prove the minimality of F_1, we also compute a subpartition $\mathcal{X} = \{X_1, \ldots, X_{q_1}, X_{q_1+1}, \ldots, X_{q_2}\}$ of V such that

$$\begin{aligned} & c_{G_1}(s, X_i) = \ell - c_G(X_i) \text{ for } i = 1, \ldots, q_1, \\ & c_{G_1}(s, X_i) = 3 - |\Gamma_G(X_i)| \text{ and } V - X_i - \Gamma_G(X_i) \neq \emptyset \text{ for } i = q_1 + 1, \ldots, q_2, \\ & \Gamma_{G_1}(s) \subseteq \cup_{X \in \mathcal{X}} X. \end{aligned} \tag{2.4}$$

Such a subpartition $\mathcal{X}$ ensures $|F_1| = \alpha_{\ell,3}(G)$. Note that $|F_1| \ge \alpha_{\ell,3}(G)$ holds since otherwise at least one of (3.1), (3.2) and (3.3) cannot hold. On the other hand $|F_1| \le \alpha_{\ell,3}(G)$ follows from the maximality of $\alpha_{\ell,3}(G)$. Therefore in order to prove Property 3.1, we present an algorithm, called ADD-EDGE*, that finds an edge set F_1 satisfying (3.1) – (3.3) and a subpartition $\mathcal{X}$ satisfying (2.4).

Algorithm ADD-EDGE*

Input: An undirected multigraph $G = (V, E)$ ($|V| \geq 4$) and an integer $\ell \geq 4$.

Output: A set of new edges F_1 with $|F_1| = \alpha_{\ell,3}(G)$ such that $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies (3.1) – (3.3).

Step 1. After adding a sufficiently large number (say, ℓ) of edges between s and each vertex $v \in V$, discard them one by one as long as all (3.1), (3.2) and (3.3) remain valid. Let $G_1' = (V \cup \{s\}, E \cup F_1')$ be the resulting multigraph where $F_1' = E_{G_1'}(s, V)$.

Step 2. For each κ-critical edge $e = (s,t) \in E_{G_1'}(s, V)$, G_1' has a κ-critical cut $T \subset V$ with $V - T - \Gamma_G(T) \neq \emptyset$ satisfying (I) or (II) in Property A.1 (note that no cut κ-critical cut T satisfies (III) since $c_{G_1'}(v) \geq \ell \geq 4$ holds for every $v \in V$ by (3.1)). A neighbor $t \in \Gamma_{G_1'}(s)$ of s is also called *κ-critical* if it is contained in a κ-critical cut T. Every κ-critical cut T of type (I) or (II) satisfies the second condition $c_{G_1'}(s,T) = 3 - |\Gamma_G(T)|$ in (2.4). Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) denote the family of all κ-critical cuts T of type (I) (resp., (II)) such that no $T' \subset T$ is of type (I) (resp., (II)).

For each edge $e = (s,u) \in F_1'$ such that $G_1' - e$ violates (3.1), there is a cut $X_u \subset V$, called *λ-critical*, such that $u \in X_u$, $c_{G_1'}(X_u) = \ell$ and $c_{G_1'}(Y) > \ell$ holds for all cuts Y with $u \in Y \subset X_u$. It is not difficult to see by (2.1) that a λ-critical cut X_u is unique for such $(s,u) \in F_1'$. Let $\mathcal{X}_1$ denotes the family of all λ-critical cuts X_u. Every cut $X \in \mathcal{X}_1$ satisfies the first condition $c_{G_1'}(s, X) = \ell - c_G(X)$ in (2.4). By definition, for any cut $X \in \mathcal{X}_1$, the set of vertices $u \in \Gamma_{G_1'}(s)$ such that the X is λ-critical with respect to u is given by $(X - \cup_{Y \in \mathcal{X}_1,\ Y \subset X} Y) \cap \Gamma_{G_1'}(s)$. Those vertices are called *free-vertices* of X in G_1'; for any edge $e = (s,u) \in E_{G_1'}(s)$ with a free-vertex u of $X \in \mathcal{X}_1$ and a vertex $v \in X$, shifting $(s,v)/e$ preserves (3.1). Notice that, for two cuts $X, X' \in \mathcal{X}_1$ with $X \subset X'$, such shifting edges within X does not change the set of free-vertices of X'.

Property B.1 *Assume that there is a family $\mathcal{Z}$ of cuts $Z \in \mathcal{X}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ with $\Gamma_{G_1'}(s) \subseteq \cup_{Z \in \mathcal{Z}} Z$ such that every two cuts $Z_1 \in \mathcal{Z} \cap \mathcal{X}_1$ and $Z_2 \in \mathcal{Z} \cap \mathcal{T}_1$ satisfies $Z_1 \cap Z_2 = \emptyset$. Then G_1' has a subpartition of V satisfying* (2.4). ☐

From this property, G_1' has a subpartition of V satisfying (2.4) unless there are two cuts $X \in \mathcal{X}_1$ and $T \in \mathcal{T}_1$ satisfying the following:

$$T \cap X \neq \emptyset \text{ and } (T - \textstyle\sum_{X_i \in \mathcal{X}_1} X_i) \cap \Gamma_{G_1'}(s) \neq \emptyset. \tag{2.5}$$

We say that a κ-critical set $T \in \mathcal{T}_1$ *violates* a cut $X \in \mathcal{X}_1$ if T and X satisfies (2.5).

Let $\mathcal{T}^{(1)}$ be the set of κ-critical sets $T \in \mathcal{T}_1$ such that their neighbors t with $\{t\} = \Gamma_{G'_1}(s) \cap T$ belong to $V - \cup_{X \in \mathcal{X}_1} X$. Let $\mathcal{X}^{(1)}$ be the set of all cuts X' in $\mathcal{X}_1$ that are violated by some $T \in \mathcal{T}^{(1)}$. If $\mathcal{X}^{(1)} = \emptyset$, then halt, outputting the current $F_1 = E_{G'_1}(s)$.

Assume that $\mathcal{X}^{(1)} \neq \emptyset$. If there is a free-vertex u of some $X' \in \mathcal{X}^{(1)}$ such that u is not κ-critical, then we execute shifting $(s, v')/(s, u)$ for a vertex $v' \in X' \cap T_1$ with $T_1 \in \mathcal{T}^{(1)}$. Then remove edge $(s, t_1) \in E_{G'_1}(s, T_1)$. In this case, we recompute a family of λ-critical sets and κ-critical sets in the resulting graph G''_1, and apply the same procedure of step 2.

Otherwise, if all free-vertices of each cut in $\mathcal{X}^{(1)}$ are κ-critical, then let $\mathcal{T}^{(2)}$ be the set of all κ-critical sets T corresponding to those κ-critical neighbors of s, and $\mathcal{X}^{(2)}$ be the set of all cuts X' in $\mathcal{X}_1 - \mathcal{X}^{(1)}$ that are violated by some $T \in \mathcal{T}^{(2)}$. If $\mathcal{X}^{(2)} = \emptyset$, then $\mathcal{X}_1 := \mathcal{X}_1 - \mathcal{X}^{(1)}$ and halt, outputting the current $E_{G'_1}(s)$.

If $\mathcal{X}^{(2)} \neq \emptyset$, then we define $\mathcal{T}^{(j)}$ and $\mathcal{X}^{(j)}$, $j = 3, 4, \ldots$, in a similar way until $\mathcal{X}^{(j')}$ becomes empty or a cut $X_{j'} \in \mathcal{X}^{(j')}$ has a non κ-critical free-vertex u^*. In the former case, $\mathcal{X}_1 := \mathcal{X}_1 - \cup_{1 \leq j < j'} \mathcal{X}^{(j)}$ and halt, outputting the current $E_{G'_1}(s)$. In the latter case, we can find a sequence of κ-critical sets $T_j \in \mathcal{T}^{(j)}$ and $X_j \in \mathcal{X}^{(j)}$, $1 \leq j \leq j'$ such that κ-critical set T_j violates X_j. For a vertex $v_j \in T_j \cap X_j$, $1 \leq j \leq j'$, we execute shifting $(s, v_{j-1})/(s, t_j)$ for each $j = 2, \ldots, j'+1$, where $t_{j'+1} = u^*$ and t_j $(1 \leq j < j'+1)$ denote the κ-critical neighbor in each T_j, and remove the edge (s, t_1) from the resulting graph. Then we choose $\mathcal{W} = \{T_1, X_1, \ldots, T_{j'}, X_{j'}\}$ such that $v_j \notin \cup_{i<j} T_i$ holds for each j. This is possible because if $v_j \in T_j \cap T_p$ holds for $j' \geq j > p \geq 1$, then we can rechoose $\mathcal{W}' = \mathcal{W} - \{X_p, T_{p+1}, X_{p+1}, \ldots, T_{j-1}, X_{j-1}, T_j\}$. Then we restart the same procedure of step 2 after recomputing a new family of λ-critical cuts and a new family of κ-critical cuts.

Property B.2 *The graph G''_1 resulting from an iteration of step 2 also satisfies* (3.1) – (3.2). □

Here we prove the correctness of the algorithm ADD-EDGE*. For this, we prove Properties B.1 and B.2.

Proof of Property B.1: Every two cuts Z_1, Z_2 with $Z_1 \in \mathcal{Z} \cap \mathcal{T}_1$ and $Z_2 \in \mathcal{Z} \cap \mathcal{T}_2$ are pairwise disjoint since we have $c_{G'_1}(s, Z_1) = 1$, $c_{G'_1}(s, Z_2) = 2$ and $|Z_2| = 1$. It is not difficult to see by (2.1) that $\mathcal{X}_1$ is laminar. Hence from the assumption, it suffices to consider the case that two cuts $Z_1, Z_2 \in \mathcal{Z} \cap \mathcal{T}_1$ intersect each other. Let $\mathcal{T}_3$ be a family of cuts $T \subseteq V$ such that $V - T - \Gamma_G(T) \neq \emptyset$, $|T| \geq 2$, $|\Gamma_G(T)| = 1$ and $c_{G'_1}(s, T) = |\Gamma_{G'_1}(s) \cap T| = 2$ hold and no $T' \subset T$ satisfies the property. If two cuts $Z_1, Z_2 \in \mathcal{Z} \cap \mathcal{T}_1$ intersect each other, then from Property A.2, we see that there is a cut $Z_3 \in \mathcal{T}_3$ with $Z_3 \subseteq Z_1 \cup Z_2$ and $\Gamma_{G'_1}(s) \cap (Z_1 \cup Z_2) \subseteq Z_3$. Now we see that every two cuts in $\mathcal{T}_3$ are pairwise disjoint from Property A.3 and the property that every cut $X \subset V$ satisfies $c_{G'_1}(X) \geq 4$ (note that every cut $T \in \mathcal{T}_5$ in Appendix A satisfies $c_{G'}(T) = 3$).

Let $\mathcal{Z}_1 \subseteq \mathcal{Z}$ be a family of sets obtained from $\mathcal{Z}$ by replacing every two intersecting cuts $Z_1, Z_2 \in \mathcal{Z} \cap \mathcal{T}_1$ with $Z_3 \subseteq Z_1 \cup Z_2$ satisfying $Z_3 \in \mathcal{T}_3$. Let $\mathcal{Z}_2 \subseteq \mathcal{Z}_1$ be a family of sets obtained from $\mathcal{Z}_1$ by removing a cut $Z \in \mathcal{Z}_1$ with $Z \subset Z'$ for some $Z' \in \mathcal{Z}_1$. From above, we see that every two cuts in $\mathcal{Z}_2$ are pairwise disjoint and we have $\Gamma_{G'_1}(s) \subseteq \cup_{Z \in \mathcal{Z}_2} Z$, and hence $\mathcal{Z}_2$ satisfies (2.4). □

To prove Property B.2, we show the next claim.

Claim B.1 *Let* $G' = (V \cup \{s\}, E \cup F)$ *be a graph satisfying* (3.2), (3.3), $G = G' - s$ *and* $F = E_{G'}(s)$. *For each* κ*-critical edge* $e = (s, v) \in E_{G'}(s)$, *a* κ*-critical cut* T_v *with* $v \in T_v$ *such that no* $T \subset T_v$ *with* $v \in T$ *is* κ*-critical is unique.*

Proof: If $|T_v| = 1$ holds, the claim clearly holds. We consider the case of $|T_v| \geq 2$. Assume that G' has two distinct κ-critical cuts T_1 and T_2 with $v \in T_1 \cap T_2$. For $i = 1, 2$, $|T_i| \geq 2$ implies that we have $T_i \in \mathcal{T}_1$ and hence $|\Gamma_G(T_i)| = 2$ and $c_{G'}(s, T_i) = 1$. From the minimality of T_1 and T_2, T_1 and T_2 intersect each other and we have $|\Gamma_G(T_1 \cap T_2)| \geq 3$. Hence (2.2) implies $|\Gamma_G(T_1 \cup T_2)| \leq 1$. From this, we see $V - (T_1 \cup T_2) - \Gamma_G(T_1 \cup T_2) = \emptyset$, since otherwise $c_{G'}(s, T_1 \cup T_2) = 1$ and $|\Gamma_G(T_1 \cup T_2)| \leq 1$ violates (3.2) in G'. Hence we have $|V - (T_1 \cup T_2)| \leq 1$. Now we have $c_{G'}(s, V - T_1 - \Gamma_G(T_1)) \geq 1$ and $c_{G'}(s, V - T_2 - \Gamma_G(T_2)) \geq 1$ by (3.2) and (3.3) in G'. By $c_{G'}(s, T_1 \cup T_2) = c_{G'}(s, T_1 \cap T_2) = 1$, we have $(V - T_1 - \Gamma_G(T_1)) \cap \Gamma_{G'}(s) \subseteq V - T_1 - T_2$ and $(V - T_2 - \Gamma_G(T_2)) \cap \Gamma_{G'}(s) \subseteq V - T_1 - T_2$. This contradicts $V - (T_1 \cup T_2) - \Gamma_G(T_1 \cup T_2) = \emptyset$ and $|V - (T_1 \cup T_2)| \leq 1$. □

Proof of Property B.2: Let G''_1 be the resulting graph after an iteration of step 2, and G'_1 be the graph immediately before the iteration. Clearly, G''_1 also satisfies (3.3), since the procedure do not remove any edge in $E_{G'_1}(s, v)$ for $\{v\} \in \mathcal{T}_2$.

We first show that G''_1 satisfies (3.1). Since (s, t_1) is not λ-critical, removal of (s, t_1) preserves (3.1). Assume that we execute shifting $(s, v_{j-1})/(s, t_j)$, $j \in \{2, \ldots, j' + 1\}$ one by one, in the order of $j = i_1, i_2, \ldots, i_{j'}$ such that $X_{i_p} \not\subset X_{i_{p+1}}$ for $1 \leq p < j'$. Assume that, after shifting $(s, v_{i_h})/(s, t_{i_h+1})$, the resulting graph G^{i_h} violates (3.1) for the first time. Let $Y \subset V$ be a cut with $c_{G^{i_h}}(Y) = \ell - 1$. Then $c_{G^{i_{h-1}}}(Y) = \ell$, $t_{i_{h+1}} \in Y$, and $v_{i_h} \notin Y$ hold, where $G^{i_{h-1}}$ denotes the graph immediately before shifting $(s, v_{i_h})/(s, t_{i_{h+1}})$ is applied. Since $\{v_{i_h}, t_{i_{h+1}}\} \subseteq X_{i_h} \in \mathcal{X}_1$, it follows $X_{i_h} - Y \neq \emptyset$ and $X_{i_h} \cap Y \neq \emptyset$. Assume that $Y - X_{i_h} \neq \emptyset$ holds. By $s \in (V \cup s) - (X_{i_h} \cup Y)$, Y and X_{i_h} cross each other. Then, by (2.1), $c_{G^{i_{h-1}}}(X_{i_h}) + c_{G^{i_{h-1}}}(Y) = c_{G^{i_{h-1}}}(X_{i_h} - Y) + c_{G^{i_{h-1}}}(Y - X_{i_h}) + 2c_{G^{i_{h-1}}}(X_{i_h} \cap Y, (V \cup s) - (X_{i_h} \cup Y))$ holds. Since $G^{i_{h-1}}$ satisfies (3.1) and $c_{G^{i_{h-1}}}(X_{i_h}) = c_{G^{i_{h-1}}}(Y) = \ell$, $c_{G^{i_{h-1}}}(X_{i_h} \cap Y, (V \cup s) - (X_{i_h} \cup Y)) = 0$ holds, contradicting $(s, t_{i_{h+1}}) \in E_{G^{i_{h-1}}}(s, X_{i_h} \cap Y)$. Thus $Y - X_{i_h} = \emptyset$. Now $t_{i_{h+1}} \in Y$, $c_{G^{i_{h-1}}}(Y) = \ell$ and $Y \subset X_{i_h}$ hold. Since $t_{i_{h+1}}$ was originally a free vertex of X_{i_h} in G'_1, we have $c_{G'_1}(Y) > \ell$. Hence for some $r < h$, there is a cut $X_{i_r} \in \mathcal{X}_1$ such that $t_{i_{r+1}} \in Y$, $v_{i_r} \in X_{i_r} - Y$ and $c_{G^{i_{r-1}}}(Y) = \ell + 1$. Since $t_{i_{r+1}}$ is a free vertex of X_{i_r}, we have $X_{i_r} \subset X_{i_h}$ by the definition of free vertices. This, however, contradicts the order of $i_1, i_2, \ldots, i_{j'}$.

We next show that G''_1 satisfies (3.2). Since u^* is not κ-critical, $G'_1 - (s, u^*)$ satisfies (3.2). Let $G^j = G^{j+1} - (s, t_j) + (s, v_j)$ for $j = 1, \ldots, j'$, where $G^{j'+1} = G'_1 - (s, u^*)$ and $G''_1 = G^1$. Assume that G^i satisfies (3.2) for $i = h + 1, \ldots, j'$ and G^h violates (3.2). Hence G^{h+1} has a κ-critical cut T' with $t_h \in T'$, $v_h \notin T'$, $|\Gamma_G(T')| = 2$ and $c_{G^{h+1}}(s, T') = c_{G^{h+1}}(s, t_h) = 1$ such that no cut $T \subset T'$ satisfies this property. Now from the construction of $\mathcal{W}$, every $v_j \notin T_h$ holds for $j \geq h+1$. Hence we have $c_{G^{h+1}}(s, T_h) = 1$ and T_h is minimal κ-critical with $t_h \in T_h$ also in G^{h+1}. Then we have $T_h \neq T'$ by $v_h \in T_h - T'$, contradicting Claim B.1. $\Box$

We show that the algorithm ADD-EDGE* can be executed in $O(n^2 m + n^3 \log n)$ time. The number of iterations of step 2 in the algorithm ADD-EDGE* is at most n, since one iteration decreases $|\Gamma_{G'_1}(s)|$.

Given G'_1, the family $\mathcal{T}_1 \cup \mathcal{T}_2$ can be computed in $O((m+n)n)$ time by applying a triconnected component algorithm [8] at most n times. The set of λ-critical edges and the laminar family $\mathcal{X}_1$ of λ-critical cuts can be computed in $O(nm + n^2 \log n)$ time [15]. Thus each iteration of step 2 can be computed in $O(n^2 m + n^3 \log n)$ time and the time complexity of the algorithm ADD-EDGE* is $O(n^2 m + n^3 \log n)$.